QUASI-NORMAL MODES AND EXPONENTIAL ENERGY DECAY FOR THE KERR-DE SITTER BLACK HOLE

SEMYON DYATLOV

Abstract. We provide a rigorous definition of quasi-normal modes for a rotating black hole. They are given by the poles of a certain meromorphic family of operators and agree with the heuristic definition in the physics literature. If the black hole rotates slowly enough, we show that these poles form a discrete subset of $\mathbb{C}$. As an application we prove that the local energy of linear waves in that background decays exponentially once orthogonality to the zero resonance is imposed.

Quasi-normal modes are the complex frequencies appearing in expansions of waves; their real part corresponds to the rate of oscillation and the nonpositive imaginary part, to the rate of decay. According to the physics literature [24] they are expected to appear in gravitational waves caused by perturbations of black holes (for more recent references and findings, see for example [8, 25, 26]). In the mathematics literature they were studied by Bachelot and Motet-Bachelot [3, 4, 5] and Sá Barreto and Zworski [31], who applied the methods of scattering theory and semiclassical analysis to the case of a spherically symmetric black hole. Quasi-normal modes were described in [31] as **resonances**; that is, poles of the meromorphic continuation of a certain family of operators; it was also proved that these poles asymptotically lie on a lattice. This was further developed by Bony and Häfner in [11], who established an expansion of the solutions of the wave equation in terms of resonant states. As a byproduct of this result, they obtained exponential decay of local energy for Schwarzschild–de Sitter. Melrose, Sá Barreto, and Vasy [30] have extended this result to more general manifolds and more general initial data.

In this paper, we employ different methods to define quasi-normal modes for the Kerr–de Sitter rotating black hole. As in [31] and [11], we use the de Sitter model; physically, this corresponds to a positive cosmological constant; mathematically, it replaces asymptotically Euclidean spatial infinity with an asymptotically hyperbolic one. Let $P_g(\omega)$, $\omega \in \mathbb{C}$, be the stationary d'Alembert–Beltrami operator of the Kerr–de Sitter metric (see Section 1 for details). It acts on functions on the space slice $M = (r_-, r_+) \times \mathbb{S}^2$. We define quasi-normal modes as poles of a certain (right) inverse $R_g(\omega)$ to $P_g(\omega)$. Because of the cylindrical symmetry of the operator $P_g(\omega)$, it leaves invariant the space $\mathcal{D}'_k$ of distributions with fixed angular momentum $k \in \mathbb{Z}$ (with respect to the axis of rotation); the inverse $R_g(\omega)$ on $\mathcal{D}'_k$ is constructed by

Theorem 1. *Let $P_g(\omega, k)$ be the restriction of $P_g(\omega)$ to $\mathcal{D}'_k$. Then there exists a family of operators*

$$R_g(\omega, k) : L^2_{\mathrm{comp}}(M) \cap \mathcal{D}'_k \to H^2_{\mathrm{loc}}(M) \cap \mathcal{D}'_k$$

meromorphic in $\omega \in \mathbb{C}$ with poles of finite rank and such that $P_g(\omega, k) R_g(\omega, k) f = f$ for each $f \in L^2_{\mathrm{comp}}(M) \cap \mathcal{D}'_k$.

Since $R_g(\omega, k)$ is meromorphic, its poles, which we call k-resonances, form a discrete set. One can then say that $\omega \in \mathbb{C}$ is a resonance, or a quasi-normal mode, if ω is a k-resonance for some $k \in \mathbb{Z}$. However, it is desirable to know that resonances form a discrete subset of $\mathbb{C}$; that is, k-resonances for different k do not accumulate near some point. Also, one wants to construct the inverse $R_g(\omega)$ that works for all values of k. For $\delta_r > 0$,[1] put

$$K_r = (r_- + \delta_r, r_+ - \delta_r),\ M_K = K_r \times \mathbb{S}^2,$$

and let 1_{M_K} be the operator of multiplication by the characteristic function of M_K (which will, based on the context, act $L^2(M_K) \to L^2(M)$ or $L^2(M) \to L^2(M_K)$). Then we are able to construct $R_g(\omega)$ on M_K for a slowly rotating black hole:

Theorem 2. *Fix $\delta_r > 0$. Then there exists $a_0 > 0$ such that if the rotation speed of the black hole satisfies $|a| < a_0$, we have the following:*

1. Every fixed compact set can only contain k-resonances for a finite number of values of k. Therefore, quasi-normal modes form a discrete subset of $\mathbb{C}$.

2. The operators $1_{M_K} R_g(\omega, k) 1_{M_K}$ define a family of operators

$$R_g(\omega) : L^2(M_K) \to H^2(M_K)$$

such that $P_g(\omega) R_g(\omega) f = f$ on M_K for each $f \in L^2(M_K)$ and $R_g(\omega)$ is meromorphic in $\omega \in \mathbb{C}$ with poles of finite rank.

As stated in Theorem 2, the operator $R_g(\omega)$ acts only on functions supported in a certain compact subset of the space slice M depending on how small a is. This is due to the fact that the operator $P_g(\omega)$ is not elliptic inside the two ergospheres located near the endpoints $r = r_\pm$. The result above can then be viewed as a construction of $R_g(\omega)$ away from the ergospheres. However, for fixed angular momentum we are able to obtain certain boundary conditions on the elements in the image of $R_g(\omega, k)$, as well as on resonant states:

Theorem 3. *Let $\omega \in \mathbb{C}$.*

1. Assume that ω is not a resonance. Take $f \in L^2_{\mathrm{comp}}(M) \cap \mathcal{D}'_k$ for some $k \in \mathbb{Z}$ and put $u = R_g(\omega, k) f \in H^2_{\mathrm{loc}}(M)$. Then u is ***outgoing*** *in the following sense: the functions*

$$v_\pm(r, \theta, \varphi) = |r - r_\pm|^{iA_\pm^{-1}(1+\alpha)(r_\pm^2 + a^2)\omega} u(r, \theta, \varphi - A_\pm^{-1}(1+\alpha) a \ln |r - r_\pm|).$$

[1] In this paper, the subscript in the constants such as δ_r, C_r, C_θ does not mean that these constants depend on the corresponding variables, such as r or θ; instead, it indicates that they are related to these variables.

are smooth near the event horizons $\{r_\pm\} \times \mathbb{S}^2$*.*

2. Assume that ω *is a resonance. Then there exists a resonant state; i.e., a nonzero solution* $u \in C^\infty(M)$ *to the equation* $P_g(\omega)u = 0$ *that is outgoing in the sense of part 1.*

The outgoing condition can be reformulated as follows. Consider the function $U = e^{-i\omega t}u$ on the spacetime $\mathbb{R} \times M$; then u is outgoing if and only if U is smooth up to the event horizons in the extension of the metric given by the Kerr-star coordinates $(t^*, r, \theta, \varphi^*)$ discussed in Section 1. This lets us establish a relation between the wave equation on Kerr-de Sitter and the family of operators $R_g(\omega)$ (Proposition 1.2). Note that here we do not follow earlier applications of scattering theory (including [11]), where spectral theory and in particular self-adjointness of P_g are used to define $R_g(\omega)$ for $\operatorname{Im}\omega > 0$ and relate it to solutions of the wave equation via Stone's formula. In the situation of the present paper, due to the lack of ellipticity of $P_g(\omega)$ inside the ergospheres, it is doubtful that P_g can be made into a self-adjoint operator; therefore, we construct $R_g(\omega)$ directly using separation of variables, cite the theory of hyperbolic equations (see Section 1) for well-posedness of the Cauchy problem for the wave equation, and prove Proposition 1.2 without any reference to spectral theory.

We now study the distribution of resonances in the slowly rotating Kerr–de Sitter case. First, we establish absense of nonzero resonances in the closed upper half-plane:

Theorem 4. *Fix* $\delta_r > 0$*. Then there exist constants* a_0 *and* C *such that if* $|a| < a_0$*, then:*

1. There are no resonances in the upper half-plane and

$$\|R_g(\omega)\|_{L^2(M_K)\to L^2(M_K)} \leq \frac{C}{|\operatorname{Im}\omega|^2},\ \operatorname{Im}\omega > 0.$$

2. There are no resonances $\omega \in \mathbb{R} \setminus 0$ *and*

$$R_g(\omega) = \frac{i(1 \otimes 1)}{4\pi(1+\alpha)(r_+^2 + r_-^2 + 2a^2)\omega} + \operatorname{Hol}(\omega),$$

where Hol *stands for a family of operators holomorphic at zero.*

Next, we use the methods of [38] and the fact that the only trapping in our situation is normally hyperbolic to get a resonance free strip:

Theorem 5. *Fix* $\delta_r > 0$ *and* $s > 0$*. Then there exist* $a_0 > 0$*,* $\nu_0 > 0$*, and* C *such that for* $|a| < a_0$*,*

$$\|R_g(\omega)\|_{L^2(M_K)\to L^2(M_K)} \leq C|\omega|^s,\ |\operatorname{Re}\omega| \geq C,\ |\operatorname{Im}\omega| \leq \nu_0.$$

Theorems 4 and 5, together with the fact that resonances form a discrete set, imply that for ν_0 small enough, zero is the only resonance in $\{\operatorname{Im}\omega \geq -\nu_0\}$. This and the presence of the global meromorphic continuation provide exponential decay of local energy:[2]

Theorem 6. *Let $(r, t^*, \theta, \varphi^*)$ be the coordinates on the Kerr–de Sitter background introduced in Section 1. Fix $\delta_r > 0$ and $s' > 0$ and assume that a is small enough. Let u be a solution to the wave equation $\Box_g u = 0$ with initial data*

$$\begin{aligned} u|_{t^*=0} &= f_0 \in H^{3/2+s'}(M) \cap \mathcal{E}'(M_K), \\ \partial_{t^*} u|_{t^*=0} &= f_1 \in H^{1/2+s'}(M) \cap \mathcal{E}'(M_K). \end{aligned} \tag{0.1}$$

Also, define the constant

$$u_0 = \frac{1+\alpha}{4\pi(r_+^2 + r_-^2 + 2a^2)} \int_{t^*=0} *(du).$$

Here $*$ denotes the Hodge star operator for the metric g (see Section 1). Then

$$\|u(t^*, \cdot) - u_0\|_{L^2(M_K)} \leq Ce^{-\nu' t^*}(\|f_0\|_{H^{3/2+s'}(M_K)} + \|f_1\|_{H^{1/2+s'}(M_K)}),\ t^* > 0,$$

for certain constants C and ν' independent of u.

For the Kerr metric, the local energy decay is polynomial as shown by Tataru and Tohaneanu [34, 33], see also the lecture notes by Dafermos and Rodnianski [13] and the references below.

Outline of the proof. The starting point of the construction of $R_g(\omega)$ is the separation of variables introduced by Teukolsky in [36]. The separation of variables techniques and the related symmetries have been used in many papers, including [2], [9], [11], [15, 16], [19, 20], [31], [37]; however, these mostly consider the case of zero cosmological constant, where other difficulties occur at zero energy and a global meromorphic continuation of the type presented here is unlikely. In our case, since the metric is invariant under axial rotation, it is enough to construct the operators $R_g(\omega, k)$ and study their behavior for large k. The operator $P_g(\omega, k)$ is next decomposed into the sum of two ordinary differential operators, P_r and P_θ (see (1.3)). The separation of variables is discussed in Section 1; the same section contains the derivation of Theorem 6 from the other theorems by the complex contour deformation method.

In the Schwarzchild–de Sitter case, P_θ is just the Laplace–Beltrami operator on the round sphere and one can use spherical harmonics to reduce the problem to studying the operator $P_r + \lambda$ for large λ. In the case $a \neq 0$, however, the operator P_θ is ω-dependent; what is more, it is no longer self-adjoint unless $\omega \in \mathbb{R}$. This raises two problems with the standard implementation of separation of variables, namely decomposing L^2 into a direct sum of the eigenspaces of P_θ. Firstly, since P_θ is not self-adjoint, we cannot automatically guarantee

[2]Recently, the author has obtained stronger exponential decay results (`arXiv:1010.5201`), as well as a more precise description of resonances and a resonance decomposition (`arXiv:1101.1260`); all of these are based on the present paper.

existence of a complete system of eigenfunctions and the corresponding eigenspaces need not be orthogonal. Secondly, the eigenvalues of P_θ are functions of ω, and meromorphy of $R_g(\omega)$ is nontrivial to show when two of these eigenvalues coincide. Therefore, instead of using the eigenspace decomposition, we write $R_g(\omega)$ as a certain contour integral (2.1) in the complex plane; the proof of meromorphy of this integral is based on Weierstrass preparation theorem. This is described in Section 2.

In Section 3, we use the separation of variables procedure to reduce Theorems 1–4 to certain facts about the radial resolvent R_r (Proposition 3.2). For fixed ω, λ, k, where $\lambda \in \mathbb{C}$ is the separation constant, R_r is constructed in Section 4 using the methods of one-dimensional scattering theory. Indeed, the radial operator P_r, after the Regge–Wheeler change of variables (4.1), is equivalent to the Schrödinger operator $P_x = D_x^2 + V_x(x)$ for a certain potential V_x (4.2). (Here $x = \pm\infty$ correspond to the event horizons.) This does not, however, provide estimates on R_r that are uniform as ω, λ, k go to infinity.

The main difficulty then is proving a uniform resolvent estimate (see (3.8)), valid for large λ and $\operatorname{Re}\lambda \gg |\operatorname{Im}\lambda| + |\omega|^2 + |ak|^2$, which in particular guarantees the convergence of the integral (2.1) and Theorem 2. A complication arises from the fact that $V_x(\pm\infty) = -\omega_\pm^2$, where $\omega_\pm$ are proportional to $(r_\pm^2 + a^2)\omega - ak$. No matter how large ω is, one can always choose k so that one of $\omega_\pm$ is small, making it impossible to use standard complex scaling[3], in the case $\omega = o(k)$, due to the lack of ellipticity of the rescaled operator at infinity. To avoid this issue, we use the analyticity of V_x and semiclassical analysis to get certain control on outgoing solutions at two distant, but fixed, points (Proposition 6.1), and then an integration by parts argument to get an L^2 bound between these two points. This is discussed in Section 6.

Finally, Section 7 contains the proof of Theorem 5. We first use the results of Sections 2–6 to reduce the problem to scattering for the Schrödinger operator P_x in the regime $\lambda = O(\omega^2)$, $k = O(\omega)$ (Proposition 7.1). In this case, we apply complex scaling to deform P_x near $x = \pm\infty$ to an elliptic operator (Proposition 7.2). We then analyse the corresponding classical flow; it is either nontrapping at zero energy, in which case the usual escape function construction (as in, for example, [27]) applies (Proposition 7.3), or has a unique maximum. In the latter case we use the methods of [38] designed to handle more general normally hyperbolic trapped sets and based on commutator estimates in a slightly exotic microlocal calculus. The argument of [38] has to be modified to use complex scaling instead of an absorbing potential near infinity (see also [38, Theorem 2]).

[3]Complex scaling originated in mathematical physics with the work of Aguilar–Combes [1], Balslev–Combes, and Simon. It has become a standard tool in chemistry for computing resonances. A microlocal approach has been developed by Helffer–Sjöstrand, and a more geometric version by Sjöstrand–Zworski [32] — see that paper for pointers to the literature. Complex scaling was reborn in numerical analysis in 1994 as the method of "perfectly matched layers" (see [7]). A nice application of the method of complex scaling to the Schwarzschild–de Sitter case is provided in [14].

It should be noted that, unlike [11] or [31], the construction of $R_g(\omega)$ in the present paper does not use the theorem of Mazzeo–Melrose [28] on the meromorphic continuation of the resolvent on spaces with asymptotically constant negative curvature (see also [21]). In [11] and [31], this theorem had to be applied to prove the existence of the meromorphic continuation of the resolvent for ω in a fixed neighborhood of zero where complex scaling could not be implemented.

Remark. The results of this paper also apply if the wave equation is replaced by the Klein–Gordon equation [23]

$$(\Box_g + m^2)u = 0,$$

where $m > 0$ is a fixed constant. The corresponding stationary operator is $P_g(\omega) + m^2\rho^2$; when restricted to the space $\mathcal{D}'_k$, it is the sum of the two operators (see Section 1)

$$P_r(\omega, k; m) = P_r(\omega, k) + m^2r^2,\ \ P_\theta(\omega; m) = P_\theta(\omega) + m^2a^2\cos^2\theta.$$

The proofs in this paper all go through in this case as well. In particular, the rescaled radial operator P_x introduced in (4.2) is a Schrödinger operator with the potential

$$V_x(x; \omega, \lambda, k; m) = (\lambda + m^2r^2)\Delta_r - (1+\alpha)^2((r^2+a^2)\omega - ak)^2.$$

Since $V_x(\pm\infty)$ is still equal to $-\omega_\pm^2$ with $\omega_\pm$ defined in (4.3), the radial resolvent can be defined as a meromorphic family of operators on the entire complex plane. Also, the term $m^2r^2\Delta_r$ in the operator P_x becomes of order $O(h^2)$ under the semiclassical rescaling and thus does not affect the arguments in Sections 6 and 7.

The only difference in the Klein–Gordon case is the absense of the resonance at zero: 1 is no longer an outgoing solution to the equation $P_xu = 0$ for $\omega = k = \lambda = 0$. Therefore, there is no u_0 term in Theorem 6, and all solutions to (0.1) decay exponentially in the compact set M_K.

1. Kerr–de Sitter metric

The Kerr–de Sitter metric is given by the formulas [12]

$$\begin{gathered}
g = -\rho^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) \\
-\frac{\Delta_\theta\sin^2\theta}{(1+\alpha)^2\rho^2}(a\,dt - (r^2+a^2)\,d\varphi)^2 \\
+\frac{\Delta_r}{(1+\alpha)^2\rho^2}(dt - a\sin^2\theta\,d\varphi)^2.
\end{gathered}$$

Here M_0 is the mass of the black hole, Λ is the cosmological constant (both of which we assume to be fixed throughout the paper), and a is the angular momentum (which we

assume to be bounded by some constant, and which is required to be small by most of our theorems);

$$\Delta_r = (r^2 + a^2)\Big(1 - \frac{\Lambda r^2}{3}\Big) - 2M_0 r,$$
$$\Delta_\theta = 1 + \alpha\cos^2\theta,$$
$$\rho^2 = r^2 + a^2\cos^2\theta,\ \ \alpha = \frac{\Lambda a^2}{3}.$$

We also put

$$A_\pm = \mp\partial_r\Delta_r(r_\pm) > 0.$$

The metric is defined for $\Delta_r > 0$; we assume that this happens on an open interval $0 < r_- < r < r_+ < \infty$. (For $a = 0$, this is true when $9\Lambda M_0^2 < 1$; it remains true if we take a small enough.) The variables $\theta \in [0,\pi]$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ are the spherical coordinates on the sphere $\mathbb{S}^2$. We define the space slice $M = (r_-, r_+) \times \mathbb{S}^2$; then the Kerr–de Sitter metric is defined on the spacetime $\mathbb{R} \times M$.

The d'Alembert–Beltrami operator of g is given by

$$\begin{aligned}\Box_g = \frac{1}{\rho^2}D_r(\Delta_r D_r) &+ \frac{1}{\rho^2\sin\theta}D_\theta(\Delta_\theta\sin\theta D_\theta)\\ &+\frac{(1+\alpha)^2}{\rho^2\Delta_\theta\sin^2\theta}(a\sin^2\theta D_t + D_\varphi)^2\\ &-\frac{(1+\alpha)^2}{\rho^2\Delta_r}((r^2+a^2)D_t + aD_\varphi)^2.\end{aligned}$$

(Henceforth we denote $D = \frac{1}{i}\partial$.) The volume form is

$$d\,\mathrm{Vol} = \frac{\rho^2\sin\theta}{(1+\alpha)^2}\,dtdrd\theta d\varphi.$$

If we replace D_t by a number $-\omega \in \mathbb{C}$, then the operator $\Box_g$ becomes equal to $P_g(\omega)/\rho^2$, where $P_g(\omega)$ is the following differential operator on M:

$$\begin{aligned}P_g(\omega) &= D_r(\Delta_r D_r) - \frac{(1+\alpha)^2}{\Delta_r}((r^2+a^2)\omega - aD_\varphi)^2\\ &+\frac{1}{\sin\theta}D_\theta(\Delta_\theta\sin\theta D_\theta) + \frac{(1+\alpha)^2}{\Delta_\theta\sin^2\theta}(a\omega\sin^2\theta - D_\varphi)^2.\end{aligned} \tag{1.1}$$

We now introduce the separation of variables for the operator $P_g(\omega)$. We start with taking Fourier series in the variable φ. For every $k \in \mathbb{Z}$, define the space

$$\mathcal{D}'_k = \{u \in \mathcal{D}' \mid (D_\varphi - k)u = 0\}. \tag{1.2}$$

This space can be considered as a subspace of $\mathcal{D}'(M)$ or of $\mathcal{D}'(\mathbb{S}^2)$ alone, and

$$L^2(M) = \bigoplus_{k\in\mathbb{Z}}(L^2(M) \cap \mathcal{D}'_k);$$

the right-hand side is the Hilbert sum of a family of closed mutually orthogonal subspaces.

Let $P_g(\omega,k)$ be the restriction of $P_g(\omega)$ to $\mathcal{D}'_k$. Then we can write

$$P_g(\omega,k)=P_r(\omega,k)+P_\theta(\omega)|_{\mathcal{D}'_k},$$

where

$$\begin{aligned}P_r(\omega,k)&=D_r(\Delta_r D_r)-\frac{(1+\alpha)^2}{\Delta_r}((r^2+a^2)\omega-ak)^2,\\ P_\theta(\omega)&=\frac{1}{\sin\theta}D_\theta(\Delta_\theta\sin\theta D_\theta)+\frac{(1+\alpha)^2}{\Delta_\theta\sin^2\theta}(a\omega\sin^2\theta-D_\varphi)^2\end{aligned}\tag{1.3}$$

are differential operators in r and (θ,φ), respectively.

Next, we introduce a modification of the Kerr-star coordinates (see [13, Section 5.1]). Following [33], we remove the singularities at $r=r_\pm$ by making the change of variables $(t,r,\theta,\varphi)\to(t^*,r,\theta,\varphi^*)$, where

$$t^*=t-F_t(r),\ \ \varphi^*=\varphi-F_\varphi(r).$$

Note that $\partial_{t^*}=\partial_t$ and $\partial_{\varphi^*}=\partial_\varphi$. In the new coordinates, the metric becomes

$$\begin{gathered}g=-\rho^2\Big(\frac{dr^2}{\Delta_r}+\frac{d\theta^2}{\Delta_\theta}\Big)\\ -\frac{\Delta_\theta\sin^2\theta}{(1+\alpha)^2\rho^2}[a\,dt^*-(r^2+a^2)\,d\varphi^*+(aF'_t(r)-(r^2+a^2)F'_\varphi(r))dr]^2\\ +\frac{\Delta_r}{(1+\alpha)^2\rho^2}[dt^*-a\sin^2\theta\,d\varphi^*+(F'_t(r)-a\sin^2\theta F'_\varphi(r))dr]^2.\end{gathered}$$

The functions F_t and F_φ are required to be smooth on (r_-,r_+) and satisfy the following conditions:

- $F_t(r)=F_\varphi(r)=0$ for $r\in K_r=[r_-+\delta_r,r_+-\delta_r]$;
- $F'_t(r)=\pm(1+\alpha)(r^2+a^2)/\Delta_r+F_{t\pm}(r)$ and $F'_\varphi(r)=\pm(1+\alpha)a/\Delta_r+F_{\varphi\pm}(r)$, where $F_{t\pm}$ and $F_{\varphi\pm}$ are smooth at $r=r_\pm$, respectively;
- for some (a-independent) constant C and all $r\in(r_-,r_+)$,

$$\frac{(1+\alpha)^2(r^2+a^2)^2}{\Delta_r}-\Delta_r F'_t(r)^2-(1+\alpha)^2a^2\geq\frac{1}{C}>0.$$

Under these conditions, the metric g in the new coordinates is smooth up to the event horizons $r=r_\pm$ and the space slices

$$M_{t_0}=\{t^*=t_0=\mathrm{const}\}\cap(\mathbb{R}\times M),\ \ t_0\in\mathbb{R},$$

are space-like. Let ν_t be the time-like normal vector field to these surfaces, chosen so that $g(\nu_t,\nu_t)=1$ and $\langle dt^*,\nu_t\rangle>0$.

We now establish a basic energy estimate for the wave equation in our setting. Let u be a real-valued function smooth in the coordinates $(t^*, r, \theta, \varphi^*)$ up to the event horizons. Define the vector field $T(du)$ by

$$T(du) = \partial_t u \nabla_g u - \frac{1}{2} g(du, du)\nu_t.$$

Since ν_t is timelike, the expression $g(T(du), \nu_t)$ is a positive definite quadratic form in du. For $t_0 \in \mathbb{R}$, define $E(t_0)(du)$ as the integral of this quadratic form over the space slice M_{t_0} with the volume form induced by the metric.

Proposition 1.1. *Take $t_1 < t_2$ and let*

$$\Omega = \{t_1 \leq t^* \leq t_2\} \times M.$$

Assume that u is smooth in Ω up to its boundary and solves the wave equation $\Box_g u = 0$ in this region. Then

$$E(t_2)(du) \leq e^{C_e(t_2 - t_1)} E(t_1)(du)$$

for some constant C_e independent of t_1 and t_2.

Proof. We use the method of [35, Proposition 2.8.1]. We apply the divergence theorem to the vector field $T(du)$ on the domain Ω. The integrals over M_{t_1} and M_{t_2} will be equal to $-E(t_1)$ and $E(t_2)$. The restriction of the metric to tangent spaces of the event horizons is nonpositive and the field ν_t is pointing outside of Ω at $r = r_\pm$; therefore, the integrals over the event horizons will be nonnegative. Finally, since $\Box_g u = 0$, one can prove that $\operatorname{div} T(du)$ is quadratic in du and thus

$$|\operatorname{div} T(du)| \leq C g(T(du), \nu_t).$$

Therefore, the divergence theorem gives

$$E(t_2) - E(t_1) \leq C \int_{t_1}^{t_2} E(t_0)\, dt_0.$$

It remains to use Gronwall's inequality. $\Box$

The geometric configuration of $\{t^* = t_1\}$, $\{t^* = t_2\}$, $\{r = r_\pm\}$, and ν_t with respect to the Lorentzian metric g used in Proposition 1.1, combined with the theory of hyperbolic equations (see [13, Proposition 3.1.1], [22, Theorem 23.2.4], or [36, Sections 2.8 and 7.7]), makes it possible to prove that for each $f_0 \in H^1(M)$, $f_1 \in L^2(M)$, there exists a unique solution

$$u(t^*, \cdot) \in C([0, \infty); H^1(M)) \cap C^1([0, \infty); L^2(M))$$

to the initial value problem

$$\Box_g u = 0,\ u|_{t^*=0} = f_0,\ \partial_{t^*} u|_{t^*=0} = f_1. \tag{1.4}$$

We are now ready to prove Theorem 6. Fix $\delta_r > 0$ and assume that a is chosen small enough so that Theorems 2–5 hold. Assume that $s' > 0$ and u is the solution to (1.4)

with $f_0 \in H^{3/2+s'} \cap \mathcal{E}'(M'_K)$ and $f_1 \in H^{1/2+s'} \cap \mathcal{E}'(M'_K)$, where M'_K is fixed and compactly contained in M_K. By finite propagation speed (see [36, Theorem 2.6.1 and Section 2.8]), there exists a function $\chi(t) \in C^\infty(0,\infty)$ independent of u and such that $\chi(t^*) = 1$ for $t^* > 1$, and for $t^* \in \operatorname{supp}(1-\chi)$, $\operatorname{supp} u(t^*, \cdot) \subset M_K$. By Proposition 1.1, we can define the Fourier-Laplace transform

$$\widehat{\chi u}(\omega) = \int e^{it^*\omega}\chi(t^*)u(t^*,\cdot)\,dt^* \in H^{3/2+s'}(M),\ \operatorname{Im}\omega > C_e.$$

Put $f = \rho^2\Box_g(\chi u) = \rho^2[\Box_g, \chi]u$; then

$$f \in H^{1/2+s'}_{\mathrm{comp}}(\mathbb{R}; L^2(M) \cap \mathcal{E}'(M_K)).$$

Therefore, one can define the Fourier-Laplace transform $\hat{f}(\omega) \in L^2 \cap \mathcal{E}'(M_K)$ for all $\omega \in \mathbb{C}$, and we have the estimate

$$\int \langle\omega\rangle^{2s'+1}\|\hat{f}(\omega)\|^2_{L^2(M)}\,d\omega \le C(\|f_0\|^2_{H^{3/2+s'}} + \|f_1\|^2_{H^{1/2+s'}}).$$

where integration is performed over the line $\{\operatorname{Im}\omega = \nu = \mathrm{const}\}$ with ν bounded.

Proposition 1.2. *We have for* $\operatorname{Im}\omega > C_e$,

$$\widehat{\chi u}(\omega)|_{M_K} = R_g(\omega)\hat{f}(\omega).$$

Proof. Without loss of generality, we may assume that $u \in C^\infty \cap \mathcal{D}'_k$ for some $k \in \mathbb{Z}$; then $R_g(\omega)\hat{f}(\omega)$ can be defined on the whole M by Theorem 1. Fix ω and put

$$\Phi(\omega) = e^{i\omega F_t(r)}\widehat{\chi u}(\omega) - R_g(\omega, k)\hat{f}(\omega) \in C^\infty(M).$$

Since $\rho^2\Box_g(\chi u) = f$, we have

$$P_g(\omega)(e^{i\omega F_t(r)}\widehat{\chi u}(\omega)) = \hat{f}(\omega);$$

therefore, $P_g(\omega)\Phi(\omega) = 0$. Note also that Φ is smooth inside M because of ellipticity of the operator $P_g(\omega)$ on $\mathcal{D}'_k$ (see [36, Section 7.4] and the last step of the proof of Theorem 1). Now, if we put

$$U(t,\cdot) = e^{-it\omega}\Phi(\omega)(\cdot),$$

then $\Box_g U = 0$ inside M_S. However, by Theorem 3, U is smooth in the $(r, t^*, \theta, \varphi^*)$ coordinates up to the event horizons and its energy grows in time faster than allowed by Proposition 1.1; therefore, $\Phi = 0$. $\square$

We now restrict our attention to the compact M_K, where in particular $t = t^*$ and $\varphi = \varphi^*$. By the Fourier Inversion Formula, for $t > 1$ and $\nu > C_e$,

$$u(t)|_{M_K} = (2\pi)^{-1}\int e^{-it(\omega+i\nu)}R_g(\omega+i\nu)\hat{f}(\omega+i\nu)\,d\omega.$$

Fix positive $s < s'$. By Theorems 4 and 5, there exists $\nu_0 > 0$ such that zero is the only resonance with $\operatorname{Im}\omega \ge -\nu_0$. Using the estimates in these theorems, we can deform the

contour of integration above to the one with $\nu = -\nu_0$. Indeed, by a density argument we may assume that $u \in C^\infty$, and in this case, $\hat{f}(\omega)$ is rapidly decreasing as $\operatorname{Re}\omega \to \infty$ for $\operatorname{Im}\omega$ fixed. We then get

$$\begin{aligned} u(t)|_{K_r} &= \frac{1+\alpha}{4\pi(r_+^2 + r_-^2 + 2a^2)}(\hat{f}(0), 1)_{L^2(K_r)} \\ &+ (2\pi)^{-1} e^{-\nu_0 t} \int e^{-it\omega} R_g(\omega - i\nu_0)\hat{f}(\omega - i\nu_0)\, d\omega. \end{aligned} \tag{1.5}$$

We find a representation of the first term above in terms of the initial data for u at time zero. We have

$$(\hat{f}(0), 1)_{L^2(K_r)} = \int_{M_K \times \mathbb{R}} \Box_g(\chi u)\, d\operatorname{Vol}.$$

Here $d\operatorname{Vol}$ is the volume form induced by g. Integrating by parts, we get

$$\int_{M_K \times \mathbb{R}} \Box_g(\chi u)\, d\operatorname{Vol} = -\int_{t \geq 0} \Box_g((1-\chi)u)\, d\operatorname{Vol} = \int_{t=0} *(du). \tag{1.6}$$

Here $*$ is the Hodge star operator induced by the metric g, with the orientation on M and $\mathbb{R} \times M$ chosen so that $*(dt)$ is positively oriented on $\{t = 0\}$.

Finally, the L^2 norm of the integral term in (1.5) can be estimated by

$$\begin{aligned} &Ce^{-\nu_0 t} \int \langle\omega\rangle^{s-s'-1/2} \|\langle\omega\rangle^{s'+1/2} \hat{f}(\omega - i\nu_0)\|_{L^2(K_r)}\, d\omega \\ &\leq Ce^{-\nu_0 t} \|\langle\omega\rangle^{s'+1/2} \hat{f}(\omega - i\nu_0)\|_{L^2_\omega(\mathbb{R}) L^2(K_r)} \\ &\leq Ce^{-\nu_0 t}(\|f_0\|_{H^{s'+3/2}} + \|f_1\|_{H^{s'+1/2}}), \end{aligned}$$

since $\langle\omega\rangle^{s-s'-1/2} \in L^2$. This proves Theorem 6.

Remark. In the original coordinates, (t, r, θ, φ), the equation $\Box_g u = 0$ has two solutions depending only on the time variable, namely, $u = 1$ and $u = t$. Even though Theorem 6 does not apply to these solutions because we only construct the family of operators $R_g(\omega)$ acting on functions on the compact set M_K, it is still interesting to see where our argument fails if $R_g(\omega)$ were well-defined on the whole M. The key fact is that our Cauchy problem is formulated in the t^* variable. Then, for $u = t$ the function $f_0 = u|_{t^*=0}$ behaves like $\log|r - r_\pm|$ near the event horizons and thus does not lie in the energy space H^1. As for $u = 1$, our theorem gives the correct form of the contribution of the zero resonance, namely, a constant; however, the value of this constant cannot be given by the integral of $*(du)$ over $t^* = 0$, as $du = 0$. This discrepancy is explained if we look closer at the last equation in (1.6); while integrating by parts, we will get a nonzero term coming from the integral of $*d(\chi(t^*))$ over the event horizons.

2. Separation of variables in an abstract setting

In this section, we construct inverses for certain families of operators with separating variables. Since the method described below can potentially be applied to other situations, we develop it abstractly, without any reference to the operators of our problem. Similar constructions have been used in other settings by Ben-Artzi–Devinatz [6] and Mazzeo–Vasy [29, Section 2].

First, let us consider a differential operator

$$P(\omega) = P_1(\omega) + P_2(\omega)$$

in the variables (x_1, x_2), where $P_1(\omega)$ is a differential operator in the variable x_1 and $P_2(\omega)$ is a differential operator in the variable x_2; ω is a complex parameter. If we take $\mathcal{H}_1$ and $\mathcal{H}_2$ to be certain L^2 spaces in the variables x_1 and x_2, respectively, then the corresponding L^2 space in the variables (x_1, x_2) is their Hilbert tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Recall that for any two bounded operators A_1 and A_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, their tensor product $A_1 \otimes A_2$ is a bounded operator on $\mathcal{H}$ and

$$\|A_1 \otimes A_2\| = \|A_1\| \cdot \|A_2\|.$$

The operator P is now written on $\mathcal{H}$ as

$$P(\omega) = P_1(\omega) \otimes 1_{\mathcal{H}_2} + 1_{\mathcal{H}_1} \otimes P_2(\omega).$$

We now wish to construct an inverse to $P(\omega)$. The method used is an infinite-dimensional generalization of the following elementary

Proposition 2.1. *Assume that A and B are two (finite-dimensional) matrices and that the matrix $A \otimes 1 + 1 \otimes B$ is invertible. (That is, no eigenvalue of A is the negative of an eigenvalue of B.) For $\lambda \in \mathbb{C}$, let $R_A(\lambda) = (A + \lambda)^{-1}$ and $R_B(\lambda) = (B - \lambda)^{-1}$. Take γ to be a bounded simple closed contour in the complex plane such that all poles of R_A lie outside of γ, but all poles of R_B lie inside γ; we assume that γ is oriented in the clockwise direction. Then*

$$(A \otimes 1 + 1 \otimes B)^{-1} = \frac{1}{2\pi i} \int_\gamma R_A(\lambda) \otimes R_B(\lambda)\, d\lambda.$$

The starting point of the method are the inverses[4]

$$R_1(\omega, \lambda) = (P_1(\omega) + \lambda)^{-1},\ R_2(\omega, \lambda) = (P_2(\omega) - \lambda)^{-1}$$

defined for $\lambda \in \mathbb{C}$. These inverses depend on two complex variables, and we need to specify their behavior near the singular points:

[4] In this section, we do not use the fact that $R_j(\omega, \lambda) = (P_j(\omega) \pm \lambda)^{-1}$, neither do we prove that $R(\omega) = P(\omega)^{-1}$. This step will be done in our particular case in the proof of Theorem 1 in the next section; in fact, R_1 will only be a right inverse to $P_1 + \lambda$. Until then, we merely establish properties of $R(\omega)$ defined by (2.1) below.

Definition 2.1. *Let $\mathcal{X}$ be any Banach space, and let W be a domain in $\mathbb{C}^2$. We say that $T(\omega,\lambda)$ is an (ω-nondegenerate) meromorphic map $W \to \mathcal{X}$ if:*

1. *$T(\omega,\lambda)$ is a (norm) holomorphic function of two complex variables with values in $\mathcal{X}$ for $(\omega,\lambda) \notin Z$, where Z is a closed subset of W, called the **divisor** of T,*
2. *for each $(\omega_0,\lambda_0) \in Z$, we can write $T(\omega,\lambda) = S(\omega,\lambda)/X(\omega,\lambda)$ near (ω_0,λ_0), where S is holomorphic with values in $\mathcal{X}$ and X is a holomorphic function of two variables (with values in $\mathbb{C}$) such that:*
   - *for each ω close to ω_0, there exists λ such that $X(\omega,\lambda) \neq 0$, and*
   - *the divisor of T is given by $\{X = 0\}$ near (ω_0,λ_0).*

Note that the definition above is stronger than the standard definition of meromorphy and it is not symmetric in ω and λ. Henceforth we will use this definition when talking about meromorphic families of operators of two complex variables. It is clear that any derivative (in ω and/or λ) of a meromorphic family is again meromorphic. Moreover, if $T(\omega,\lambda)$ is meromorphic and we fix ω, then T is a meromorphic family in λ.

If $\mathcal{X}$ is the space of all bounded operators on some Hilbert space (equipped with the operator norm), then it makes sense to talk about having poles of finite rank:

Definition 2.2. *Let $\mathcal{H}$ be a Hilbert space and let $T(\omega,\lambda)$ be a meromorphic family of operators on $\mathcal{H}$ in the sense of Definition 2.1. For (ω_0,λ_0) in the divisor of T, consider the decomposition*

$$T(\omega_0,\lambda) = T_H(\lambda) + \sum_{j=1}^{N} \frac{T_j}{(\lambda-\lambda_0)^j}.$$

*Here T_H is holomorphic near λ_0 and T_j are some operators. We say that T has **poles of finite rank** if every operator T_j in the above decomposition of every ω-derivative of T near every point in the divisor is finite-dimensional.*

One can construct meromorphic families of operators with poles of finite rank by using the following generalization of Analytic Fredholm Theory:

Proposition 2.2. *Assume that $T(\omega,\lambda) : \mathcal{H}_1 \to \mathcal{H}_2$, $(\omega,\lambda) \in \mathbb{C}^2$, is a holomorphic family of Fredholm operators, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are some Hilbert spaces. Moreover, assume that for each ω, there exists λ such that the operator $T(\omega,\lambda)$ is invertible. Then $T(\omega,\lambda)^{-1}$ is a meromorphic family of operators $\mathcal{H}_2 \to \mathcal{H}_1$ with poles of finite rank. (The divisor is the set of all points where T is not invertible.)*

Proof. We can use the proof of the standard Analytic Fredholm Theory via Grushin problems, see for example [17, Theorem C.3]. □

We now go back to constructing the inverse to $P(\omega)$. We assume that

(A) $R_j(\omega,\lambda)$, $j = 1,2$, are two families of bounded operators on $\mathcal{H}_j$ with poles of finite rank. Here ω lies in a domain $\Omega \subset \mathbb{C}$ and $\lambda \in \mathbb{C}$.

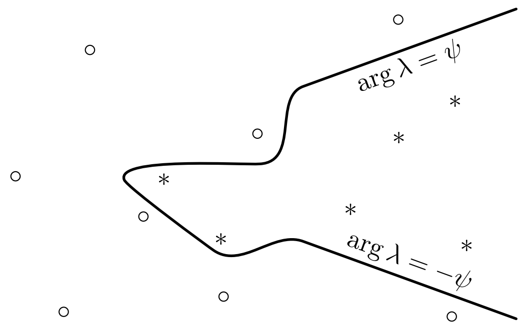

FIGURE 1. An admissible contour. The poles of R_1 are denoted by circles and the poles of R_2 are denoted by asterisks.

We want to integrate the tensor product $R_1 \otimes R_2$ in λ over a contour γ that separates the sets of poles of $R_1(\omega, \cdot)$ and $R_2(\omega, \cdot)$. Let Z_j be the divisor of R_j. We call a point ω **regular** if the sets $Z_1(\omega)$ and $Z_2(\omega)$ given by

$$Z_j(\omega) = \{\lambda \in \mathbb{C} \mid (\omega, \lambda) \in Z_j\}$$

do not intersect. The behavior of the contour γ at infinity is given by the following

Definition 2.3. *Let $\psi \in (0, \pi)$ be a fixed angle, and let ω be a regular point. A smooth simple contour γ on $\mathbb{C}$ is called **admissible** (at ω) if:*

- *outside of some compact subset of $\mathbb{C}$, γ is given by the rays $\arg\lambda = \pm\psi$, and*
- *γ separates $\mathbb{C}$ into two regions, Γ_1 and Γ_2, such that sufficiently large positive real numbers lie in Γ_2, and $Z_j(\omega) \subset \Gamma_j$ for $j = 1, 2$.*

(Henceforth, we assume that $\arg\lambda \in [-\pi, \pi]$. The contour γ and the regions Γ_j are allowed to have several connected components.)

Existence of admissible contours and convergence of the integral is guaranteed by the following condition:

(B) For any compact $K_\omega \subset \Omega$, there exist constants C and R such that for $\omega \in K_\omega$ and $|\lambda| \geq R$,
  - for $|\arg\lambda| \leq \psi$, we have $(\omega, \lambda) \notin Z_1$ and $\|R_1(\omega, \lambda)\| \leq C/|\lambda|$, and
  - for $|\arg\lambda| \geq \psi$, we have $(\omega, \lambda) \notin Z_2$ and $\|R_2(\omega, \lambda)\| \leq C/|\lambda|$.

It follows from (B) that there exist admissible contours at every regular point. Take a regular point ω, an admissible contour γ at ω, and define

$$R(\omega) = \frac{1}{2\pi i} \int_\gamma R_1(\omega, \lambda) \otimes R_2(\omega, \lambda)\, d\lambda. \tag{2.1}$$

Here the orientation of γ is chosen so that Γ_1 always stays on the left. The integral above converges and is independent of the choice of an admissible contour γ. Moreover, the set of regular points is open and R is holomorphic on this set. (We may represent $R(\omega)$ as a

locally uniform limit of the integral over the intersection of γ with a ball whose radius goes to infinity.)

The main result of this section is

Proposition 2.3. *Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is their Hilbert tensor product. Let $R_1(\omega, \lambda)$ and $R_2(\omega, \lambda)$ be two families of bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, for $\omega \in \Omega \subset \mathbb{C}$ and $\lambda \in \mathbb{C}$. Assume that R_1 and R_2 satisfy assumptions (A)–(B) and the nondegeneracy assumption*

(C) *The set Ω_R of all regular points is nonempty.*

Then the set of all non-regular points is discrete and the operator $R(\omega)$ defined by (2.1) *is meromorphic in $\omega \in \Omega$ with poles of finite rank.*

The rest of this section contains the proof of Proposition 2.3. First, let us establish a normal form for meromorphic decompositions of families in two variables:

Proposition 2.4. *Let $T(\omega, \lambda)$ be meromorphic (with values in some Banach space) and assume that (ω_0, λ_0) lies in the divisor of T. Then we can write near (ω_0, λ_0)*

$$T(\omega, \lambda) = \frac{S(\omega, \lambda)}{Q(\omega, \lambda)},$$

where S is holomorphic and Q is a monic polynomial in λ of degree N and coefficients holomorphic in ω; moreover, $Q(\omega_0, \lambda) = (\lambda - \lambda_0)^N$. The divisor of T coincides with the set of zeroes of Q near (ω_0, λ_0).

Proof. Follows from Definition 2.1 and Weierstrass Preparation Theorem. □

Proposition 2.5. *Assume that $Q_j(\omega, \lambda)$, $j = 1, 2$, are two monic polynomials in λ of degrees N_j with coefficients holomorphic in ω near ω_0. Assume also that for some ω, Q_1 and Q_2 are coprime as polynomials. Then there exist unique polynomials p_1 and p_2 of degree no more than $N_2 - 1$ and $N_1 - 1$, respectively, with coefficients meromorphic in ω and such that*

$$1 = p_1 Q_1 + p_2 Q_2$$

when p_1 and p_2 are well-defined.

Proof. The $N_1 + N_2$ coefficients of p_1 and p_2 solve a system of $N_1 + N_2$ linear equations with fixed right-hand side and the matrix $A(\omega)$ depending holomorphically on ω. If ω is chosen so that Q_1 and Q_2 are coprime, then the system has a unique solution; therefore, the determinant of $A(\omega)$ is not identically zero. The proposition then follows from Cramer's Rule. □

We are now ready to prove that $R(\omega)$ is meromorphic. It suffices to show that for each $\omega_0 \notin \Omega_R$ lying in the closure $\overline{\Omega_R}$, ω_0 is an isolated non-regular point and $R(\omega)$ has a meromorphic decomposition at ω_0 with finite-dimensional principal part. Indeed, in this

case $\overline{\Omega_R}$ is open; since it is closed and nonempty by (C), we have $\overline{\Omega_R} = \Omega$ and the statement above applies to each ω_0.

Let $Z_1(\omega_0) \cap Z_2(\omega_0) = \{\lambda_1, \dots, \lambda_m\}$. We choose a ball Ω_0 centered at ω_0 and disjoint balls U_l centered at λ_l such that:

- for $\omega \in \Omega_0$, the set $Z_1(\omega) \cap Z_2(\omega)$ is covered by balls U_l and the set $Z_1(\omega) \cup Z_2(\omega)$ does not intersect the circles ∂U_l;
- for $\omega \in \Omega_0$ and $\lambda \in U_l$, we have $R_j = S_{jl}/Q_{jl}$, where S_{jl} are holomorphic and Q_{jl} are monic polynomials in λ of degree N_{jl} with coefficients holomorphic in ω, and $Q_{jl}(\omega_0, \lambda) = (\lambda - \lambda_l)^{N_{jl}}$;
- for $\omega \in \Omega_0$, the set of all roots of $Q_{jl}(\omega, \cdot)$ coincides with $Z_j(\omega) \cap U_l$;
- there exists a contour γ_0 that does not intersect any U_l and is admissible for any $\omega \in \Omega_0$ with respect to the sets $Z_j(\omega) \setminus \cup U_l$ in place of $Z_j(\omega)$; moreover, each ∂U_l lies in the region Γ_1 with respect to γ_0 (see Definition 2.3).

Let us assume that $\omega \in \Omega_0$ is regular. (Such points exist since ω_0 lies in the closure of Ω_R.) For every l, the polynomials $Q_{1l}(\omega, \lambda)$ and $Q_{2l}(\omega, \lambda)$ are coprime; we find by Proposition 2.5 unique polynomials $p_{1l}(\omega, \lambda)$ and $p_{2l}(\omega, \lambda)$ such that

$$1 = p_{1l} Q_{1l} + p_{2l} Q_{2l}$$

and $\deg p_{1l} < N_{2l}$, $\deg p_{2l} < N_{1l}$. The converse is also true: if all coefficients of p_{1l} and p_{2l} are holomorphic at some point ω for all l, then ω is a regular point. It follows immediately that ω_0 is an isolated non-regular point.

To obtain the meromorphic expansion of $R(\omega)$ near ω_0, let us take a regular point $\omega \in \Omega_0$ and an admissible contour $\gamma = \gamma_0 + \dots + \gamma_m$, where γ_0 is the ω-independent contour defined above and each γ_l is a contour lying in U_l. The integral over γ_0 is holomorphic near ω_0, while

$$\begin{aligned} &\int_{\gamma_l} R_1(\omega, \lambda) \otimes R_2(\omega, \lambda)\, d\lambda \\ &= \int_{\gamma_l} S_{1l}(\omega, \lambda) \otimes S_{2l}(\omega, \lambda) \left(\frac{p_{1l}(\omega, \lambda)}{Q_{2l}(\omega, \lambda)} + \frac{p_{2l}(\omega, \lambda)}{Q_{1l}(\omega, \lambda)} \right) d\lambda \\ &= \int_{\partial U_l} p_{1l} S_{1l} \otimes R_2\, d\lambda = \sum_{j=0}^{N_{2l}-1} p_{1lj}(\omega) \int_{\partial U_l} (\lambda - \lambda_l)^j S_{1l} \otimes R_2\, d\lambda. \end{aligned}$$

Here $p_{1lj}(\omega)$ are the coefficients of p_{1l} as a polynomial of $\lambda - \lambda_l$; they are meromorphic in ω and the rest is holomorphic in $\omega \in \Omega_0$.

It remains to prove that R has poles of finite rank. It suffices to show that every derivative in ω of the last integral above at $\omega = \omega_0$ has finite rank. Each of these, in turn, is a finite linear combination of

$$\int_{\partial U_l} (\lambda - \lambda_l)^j \partial_\omega^a S_{1l}(\omega_0, \lambda) \otimes \partial_\omega^b R_2(\omega_0, \lambda)\, d\lambda.$$

However, since $\partial_\omega^a S_{1l}(\omega_0,\lambda)$ is holomorphic in $\lambda \in U_l$, only the principal part of the Laurent decomposition of $\partial_\omega^b R_2(\omega_0,\lambda)$ at $\lambda = \lambda_l$ will contribute to this integral; therefore, the image of each operator in the principal part of Laurent decomposition of $R(\omega)$ at ω_0 lies in $\mathcal{H}_1 \otimes V_2$, where V_2 is a certain finite-dimensional subspace of $\mathcal{H}_2$. It remains to show that each of these images also lies in $V_1 \otimes \mathcal{H}_2$, where V_1 is a certain finite-dimensional subspace of $\mathcal{H}_1$. This is done by the same argument, using the fact that

$$\int_{\partial U_l - \gamma_l} R_1(\omega,\lambda) \otimes R_2(\omega,\lambda)\, d\lambda$$

can be written in terms of p_{2l} and $R_1 \otimes S_{2l}$ and the integral over ∂U_l is holomorphic at ω_0. The proof of Proposition 2.3 is finished.

3. Construction of $R_g(\omega)$

As we saw in the previous section, one can deduce the existence of an inverse to $P_g = P_r + P_\theta$ and its properties from certain properties of the inverses to $P_r + \lambda$ and $P_\theta - \lambda$ for $\lambda \in \mathbb{C}$. We start with the latter. For $a = 0$, P_θ is the (negative) Laplace–Beltrami operator for the round metric on $\mathbb{S}^2$; therefore, its eigenvalues are given by $\lambda = l(l+1)$ for $l \in \mathbb{Z}$, $l \geq 0$. Moreover, if $\mathcal{D}'_k$ is the space defined in (1.2) and there is an eigenfunction of $P_\theta|_{\mathcal{D}'_k}$ with eigenvalue $l(l+1)$, then $l \geq k$. These observations can be generalized to our case:

Proposition 3.1. *There exists a two-sided inverse*

$$R_\theta(\omega,\lambda) = (P_\theta(\omega) - \lambda)^{-1} : L^2(\mathbb{S}^2) \to H^2(\mathbb{S}^2),\ (\omega,\lambda) \in \mathbb{C}^2,$$

with the following properties:

1. $R_\theta(\omega,\lambda)$ is meromorphic with poles of finite rank in the sense of Definition 2.2 and it has the following meromorphic decomposition at $\omega = \lambda = 0$:

$$R_\theta(\omega,\lambda) = \frac{S_{\theta 0}(\omega,\lambda)}{\lambda - \lambda_\theta(\omega)} \tag{3.1}$$

where $S_{\theta 0}$ and λ_θ are holomorphic in a-independent neighborhoods of zero and

$$S_{\theta 0}(0,0) = -\frac{1 \otimes 1}{4\pi},\ \lambda_\theta(\omega) = O(|\omega|^2).$$

2. There exists a constant C_θ such that

$$\|R_\theta(\omega,\lambda)\|_{L^2(\mathbb{S}^2)\cap\mathcal{D}'_k \to L^2(\mathbb{S}^2)} \leq \frac{C_\theta}{|k|^2} \text{ for } |\lambda| \leq k^2/2,\ |k| \geq C_\theta |a\omega|. \tag{3.2}$$

and

$$\begin{gathered}\|R_\theta(\omega,\lambda)\|_{L^2(\mathbb{S}^2)\cap\mathcal{D}'_k \to L^2(\mathbb{S}^2)} \leq \frac{2}{|\operatorname{Im}\lambda|} \\ \text{for } |\operatorname{Im}\lambda| > C_\theta |a|(|a\omega| + |k|)|\operatorname{Im}\omega|.\end{gathered} \tag{3.3}$$

3. For every $\psi > 0$, there exists a constant C_ψ such that

$$\|R_\theta(\omega,\lambda)\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq \frac{C_\psi}{|\lambda|} \text{ for } |\arg\lambda| \geq \psi,\ |\lambda| \geq C_\psi |a\omega|^2. \tag{3.4}$$

Proof. 1. Recall (1.3) that $P_\theta(\omega)$ is a holomorphic family of elliptic second order differential operators on the sphere. Therefore, for each λ, the operator $P_\theta(\omega) - \lambda : H^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is Fredholm (see for example [36, Section 7.10]). By Proposition 2.2, $R_g(\omega,\lambda)$ is a meromorphic family of operators $L^2 \to H^2$.

We now obtain a meromorphic decomposition for R_θ near zero using the framework of Grushin problems [17, Appendix C]. Let $i_1 : \mathbb{C} \to L^2(\mathbb{S}^2)$ be the operator of multiplicaton by the constant function 1 and $\pi_1 : H^2(\mathbb{S}^2) \to \mathbb{C}$ be the operator mapping every function to its integral over the standard measure on the round sphere. Consider the operator $A(\omega,\lambda) : H^2 \oplus \mathbb{C} \to L^2 \oplus \mathbb{C}$ given by

$$A(\omega,\lambda) = \begin{pmatrix} P_\theta(\omega) - \lambda & i_1 \\ \pi_1 & 0 \end{pmatrix}.$$

The kernel and cokernel of $P_\theta(0)$ are both one-dimensional and spanned by 1, since this is the Laplace–Beltrami operator for a certain Riemannian metric on the sphere. (Indeed, by ellipticity these spaces consist of smooth functions; by self-adjointness, the kernel and cokernel coincide; one can then apply Green's formula [36, (2.4.8)] to an element of the kernel and itself.) Therefore [17, Theorem C.1], the operator $B(\omega,\lambda) = A(\omega,\lambda)^{-1}$ is well-defined at $(0,0)$; then it is well-defined for (ω,λ) in an a-independent neighborhood of zero. We write

$$B(\omega,\lambda) = \begin{pmatrix} B_{11}(\omega,\lambda) & B_{12}(\omega,\lambda) \\ B_{21}(\omega,\lambda) & B_{22}(\omega,\lambda) \end{pmatrix}.$$

Now, by Schur's complement formula we have near $(0,0)$,

$$R_\theta(\omega,\lambda) = B_{11}(\omega,\lambda) - B_{12}(\omega,\lambda) B_{22}(\omega,\lambda)^{-1} B_{21}(\omega,\lambda).$$

However, $B_{22}(\omega,\lambda)$ is a holomorphic function of two variables, and we can find

$$B_{22}(\omega,\lambda) = \frac{\lambda}{4\pi} + O(|\omega|^2 + |\lambda|^2).$$

(The ω-derivative vanishes at zero since $\partial_\omega P_\omega(0)|_{\mathcal{D}'_0} = 0$. To compute the λ-derivative, we use that $B_{12}(0,0) = i_1/4\pi$ and $B_{21}(0,0) = \pi_1/4\pi$.) The decomposition (3.1) now follows by Weierstrass Preparation Theorem.

2. We have $P_\theta(\omega) = P_\theta(0) + P'_\theta(\omega)$, where

$$P'_\theta(\omega) = \frac{(1+\alpha)^2 a\omega}{\Delta_\theta}(-2D_\varphi + a\omega \sin^2\theta)$$

is a first order differential operator and

$$P_\theta(0) = \frac{1}{\sin\theta} D_\theta(\Delta_\theta \sin\theta D_\theta) + \frac{(1+\alpha)^2}{\Delta_\theta \sin^2\theta} D_\varphi^2 : H^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$$

satisfies $P_\theta(0) \geq k^2$ on $\mathcal{D}'_k$; therefore, if $u \in H^2(\mathbb{S}^2) \cap \mathcal{D}'_k$, then

$$\|u\|_{L^2} \leq \frac{\|(P_\theta(0) - \lambda)u\|_{L^2}}{d(\lambda, k^2 + \mathbb{R}^+)}.$$

Since

$$\|P'_\theta(\omega)\|_{L^2(\mathbb{S}^2)\cap\mathcal{D}'_k \to L^2(\mathbb{S}^2)} \leq 2(1+\alpha)^2 |a\omega|(|a\omega| + |k|),$$

we get

$$\|u\|_{L^2} \leq \frac{\|(P_\theta(\omega) - \lambda)u\|_{L^2}}{d(\lambda, k^2 + \mathbb{R}^+) - C_1|a\omega|(|a\omega| + |k|)}, \tag{3.5}$$

provided that the denominator is positive. Here C_1 is a global constant.

Now, if $|\lambda| \leq k^2/2$, then $d(\lambda, k^2 + \mathbb{R}^+) \geq k^2/2$ and

$$d(\lambda, k^2 + \mathbb{R}^+) - C_1|a\omega|(|a\omega| + |k|) \geq \frac{k^2}{4} \text{ for } |k| \geq 8(1 + C_1)|a\omega|;$$

together with (3.5), this proves (3.2).

To prove (3.3), introduce

$$\operatorname{Im} P_\theta(\omega) = \frac{1}{2}(P_\theta(\omega) - P_\theta(\omega)^*) = \frac{2(1+\alpha)^2}{\Delta_\theta} a \operatorname{Im}\omega (a \operatorname{Re}\omega \sin^2\theta - D_\varphi);$$

we have

$$\|\operatorname{Im} P_\theta(\omega)\|_{L^2(\mathbb{S}^2)\cap\mathcal{D}'_k \to L^2(\mathbb{S}^2)} \leq 2(1+\alpha)^2 |a \operatorname{Im}\omega|(|a\omega| + |k|).$$

However, for $u \in H^2(\mathbb{S}^2) \cap \mathcal{D}'_k$,

$$\begin{aligned}\|(P_\theta(\omega) - \lambda)u\| \cdot \|u\| &\geq |\operatorname{Im}((P_\theta(\omega) - \lambda)u, u)| \geq |\operatorname{Im}\lambda| \cdot \|u\|^2 - |(\operatorname{Im} P_\theta(\omega)u, u)| \\ &\geq (|\operatorname{Im}\lambda| - 2(1+\alpha)^2|a|(|a\omega| + |k|)|\operatorname{Im}\omega|)\|u\|^2\end{aligned}$$

and we are done if $C_\theta \geq 4(1+\alpha)^2$.

3. If $|\arg\lambda| \geq \psi$, then $d(\lambda, k^2 + \mathbb{R}^+) \geq (k^2 + |\lambda|)/C_2$; here C_2 is a constant depending on ψ. We have then

$$d(\lambda, k^2 + \mathbb{R}^+) - C_1|a\omega|(|a\omega| + |k|) \geq \frac{1}{C_2}|\lambda| - C_3|a\omega|^2$$

for some constant C_3, and we are done by (3.5). $\square$

The analysis of the radial operator P_r is more complicated. In Sections 4–6, we prove

Proposition 3.2. *There exists a family of operators*

$$R_r(\omega,\lambda,k):L^2_{\mathrm{comp}}(r_-,r_+)\to H^2_{\mathrm{loc}}(r_-,r_+),\ (\omega,\lambda)\in\mathbb{C}^2,$$

with the following properties:

1. For each $k\in\mathbb{Z}$, $R_r(\omega,\lambda,k)$ is meromorphic with poles of finite rank in the sense of Definition 2.2, and $(P_r(\omega,k)+\lambda)R_r(\omega,\lambda,k)f=f$ for each $f\in L^2_{\mathrm{comp}}(r_-,r_+)$. Also, for $k=0$, R_r admits the following meromorphic decomposition near $\omega=\lambda=0$:

$$R_r(\omega,\lambda,0)=\frac{S_{r0}(\omega,\lambda)}{\lambda-\lambda_r(\omega)},\tag{3.6}$$

where S_{r0} and λ_r are holomorphic in a-independent neighborhoods of zero and

$$S_{r0}(0,0)=\frac{1\otimes 1}{r_+-r_-},$$

$$\lambda_r(\omega)=\frac{i(1+\alpha)(r_+^2+r_-^2+2a^2)}{r_+-r_-}\omega+O(|\omega|^2).$$

2. Take $\delta_r>0$. Then there exist $\psi>0$ and C_r such that for

$$|\lambda|\geq C_r,\ |\arg\lambda|\leq\psi,\ |ak|^2\leq|\lambda|/C_r,\ |\omega|^2\leq|\lambda|/C_r,\tag{3.7}$$

(ω,λ,k) is not a pole of R_r and we have

$$\|1_{K_r}R_r(\omega,\lambda,k)1_{K_r}\|_{L^2\to L^2}\leq\frac{C_r}{|\lambda|}.\tag{3.8}$$

Also, there exists $\delta_{r0}>0$ such that, if $K_+=[r_+-\delta_{r0},r_+]$ and $K_-=[r_-,r_-+\delta_{r0}]$, then for each N there exists a constant C_N such that under the conditions (3.7), we have

$$\|1_{K_\pm}|r-r_\pm|^{iA_\pm^{-1}(1+\alpha)((r_\pm^2+a^2)\omega-ak)}R_r(\omega,\lambda,k)1_{K_r}\|_{L^2\to C^N(K_\pm)}\leq\frac{C_N}{|\lambda|^N}.\tag{3.9}$$

3. There exists a constant C_ω such that $R_r(\omega,\lambda,k)$ does not have any poles for real λ and real ω with $|\omega|>C_\omega|ak|$.

4. Assume that R_r has a pole at (ω,λ,k). Then there exists a nonzero solution $u\in C^\infty(r_-,r_+)$ to the equation $(P_r(\omega,k)+\lambda)u=0$ such that the functions

$$|r-r_\pm|^{iA_\pm^{-1}(1+\alpha)((r_\pm^2+a^2)\omega-ak)}u(r)$$

are real analytic at $r_\pm$, respectively.

5. Take $\delta_r>0$. Then there exists $C_{1r}>0$ such that for

$$\operatorname{Im}\omega>0,\ |ak|\leq|\omega|/C_{1r},\ |\operatorname{Im}\lambda|\leq|\omega|\cdot\operatorname{Im}\omega/C_{1r},\ \operatorname{Re}\lambda\geq-|\omega|^2/C_{1r},\tag{3.10}$$

(ω,λ,k) is not a pole of R_r and we have

$$\|1_{K_r}R_r(\omega,\lambda,k)1_{K_r}\|_{L^2\to L^2}\leq\frac{C_{1r}}{|\omega|\operatorname{Im}\omega}.\tag{3.11}$$

Given these two propositions, we can now prove Theorems 1–4:

Proof of Theorem 1. Take $k \in \mathbb{Z}$ and an arbitrary $\delta_r > 0$; put $\mathcal{H}_1 = L^2(K_r)$, $\mathcal{H}_2 = L^2(\mathbb{S}^2) \cap \mathcal{D}'_k$, $R_1(\omega, \lambda) = R_r(\omega, \lambda, k)$, and $R_2(\omega, \lambda) = R_\theta(\omega, \lambda)|_{\mathcal{D}'_k}$; finally, let the angle ψ of admissible contours at infinity be chosen as in Proposition 3.2. We now apply Proposition 2.3. Condition (A) follows from the first parts of Propositions 3.1 and 3.2. Condition (B) follows from (3.4) and part 2 of Proposition 3.2. Finally, condition (C) holds because every $\omega \in \mathbb{R}$ with $|\omega| > C_\omega|ak|$, where C_ω is the constant from part 3 of Proposition 3.2, is regular. Indeed, $P_\theta(\omega)$ is self-adjoint and thus has only real eigenvalues. Now, by Proposition 2.3 we can use (2.1) to define $R_g(\omega, k)$ as a meromorphic family of operators on $L^2(M_K) \cap \mathcal{D}'_k$ with poles of finite rank. This can be done for any $\delta_r > 0$; therefore, $R_g(\omega, k)$ is defined as an operator $L^2_{\mathrm{comp}}(M) \cap \mathcal{D}'_k \to L^2_{\mathrm{loc}}(M) \cap \mathcal{D}'_k$.

Let us now prove that $P_g(\omega, k) R_g(\omega, k) f = f$ in the sense of distributions for each $f \in L^2_{\mathrm{comp}}$. We will use the method of Proposition 2.1. Assume that ω is a regular point, so that $R_g(\omega, k)$ is well-defined. By analyticity, we can further assume that ω is real, so that $L^2(\mathbb{S}^2) \cap \mathcal{D}'_k$ has an orthonormal basis of eigenfunctions of $P_\theta(\omega)$. Then it suffices to prove that

$$I = (R_g(\omega, k)(f_r(r) f_\theta(\theta, \varphi)), P_g(\omega)(h_r(r) h_\theta(\theta, \varphi))) = (f_r, h_r) \cdot (f_\theta, h_\theta),$$

where $f_r, h_r \in C_0^\infty(r_-, r_+)$, $h_\theta \in C^\infty(\mathbb{S}^2) \cap \mathcal{D}'_k$, and $f_\theta \in \mathcal{D}'_k$ satisfies

$$P_\theta(\omega) f_\theta = \lambda_0 f_\theta,\ \ \lambda_0 \in \mathbb{R}.$$

Take an admissible contour γ; then

$$\begin{aligned} I = \frac{1}{2\pi i} \int_\gamma & (R_r(\omega, \lambda, k) f_r, P_r(\omega, k) h_r) \cdot (R_\theta(\omega, \lambda) f_\theta, h_\theta) \\ & + (R_r(\omega, \lambda, k) f_r, h_r) \cdot (R_\theta(\omega, \lambda) f_\theta, P_\theta(\omega) h_\theta) \, d\lambda. \end{aligned}$$

However,

$$R_\theta(\omega, \lambda) f_\theta = \frac{f_\theta}{\lambda_0 - \lambda}.$$

It then follows from condition (B) that we can replace γ by a closed bounded contour γ' which contains λ_0, but no poles of R_r. (To obtain γ', we can cut off the infinite ends of γ sufficiently far and connect the resulting two endpoints by the arc $-\psi \le \arg \lambda \le \psi$; the

integral over the arc can be made arbitrarily small.) Then

$$\begin{aligned} I = \frac{1}{2\pi i}\int_{\gamma'} ((1-\lambda R_r(\omega,\lambda,k))f_r,h_r)\cdot(R_\theta(\omega,\lambda)f_\theta,h_\theta) \\ +(R_r(\omega,\lambda,k)f_r,h_r)\cdot((1+\lambda R_\theta(\omega,\lambda))f_\theta,h_\theta)\,d\lambda \\ = \frac{1}{2\pi i}\int_{\gamma'}(f_r,h_r)\cdot(R_\theta(\omega,\lambda)f_\theta,h_\theta)+(R_r(\omega,\lambda,k)f_r,h_r)\cdot(f_\theta,h_\theta)\,d\lambda \\ = \frac{1}{2\pi i}\int_{\gamma'}\frac{(f_r,h_r)(f_\theta,h_\theta)}{\lambda_0-\lambda}\,d\lambda = (f_r,h_r)(f_\theta,h_\theta), \end{aligned}$$

which finishes the proof.

Finally, the operator $P_g(\omega,k)$ is the restriction to $\mathcal{D}'_k$ of the elliptic differential operator on M obtained from $P_g(\omega)$ by replacing D_φ by k in the second term of (1.1). Therefore, by elliptic regularity (see for example [36, Section 7.4]) the operator $R_g(\omega,k)$ acts into H^2_{loc}. □

Next, Theorem 2 follows from Theorem 1, the fact that the operator $P_g(\omega)$ is elliptic on M_K for small a (to get H^2 regularity instead of L^2), and the following estimate on $R_g(\omega,k)$ for large values of k:

Proposition 3.3. *Fix $\delta_r>0$. Then there exists $a_0>0$ and a constant C_k such that for $|a|<a_0$ and $|k|\geq C_k(1+|\omega|)$, ω is not a pole of $R_g(\cdot,k)$ and we have*

$$\|1_{M_K}R_g(\omega,k)1_{M_K}\|_{L^2\cap\mathcal{D}'_k\to L^2}\leq\frac{C_k}{|k|^2}. \tag{3.12}$$

Proof. Let ψ,C_r be the constants from part 2 of Proposition 3.2 and C_θ,C_ψ be the constants from Proposition 3.1. Put $\lambda_0=k^2/3$; if C_k is large enough, then

$$|k|>1+C_\theta|a\omega|,\ \lambda_0>C_\psi|a\omega|^2+C_r(1+|\omega|^2).$$

Take the contour γ consisting of the rays $\{\arg\lambda=\pm\psi,|\lambda|\geq\lambda_0\}$ and the arc $\{|\lambda|=\lambda_0,\ |\arg\lambda|\leq\psi\}$. By (3.2) and (3.4), all poles of R_θ lie inside γ (namely, in the region $\{|\lambda|\geq\lambda_0,\ |\arg\lambda|\leq\psi\}$), and

$$\|R_\theta(\omega,\lambda)\|_{L^2(\mathbb{S}^2)\cap\mathcal{D}'_k\to L^2(\mathbb{S}^2)}\leq\frac{C}{|\lambda|} \tag{3.13}$$

for each λ on γ. Now, suppose that $|a|<a_0=(3C_r)^{-1/2}$; then (3.7) is satisfied inside γ and (3.12) follows from (2.1), (3.8), and (3.13). □

Proof of Theorem 3. 1. Fix $\delta_r>0$ such that $\operatorname{supp}f\subset M_K$. Take an admissible contour γ; then by (2.1) and the fact that the considered functions are in $\mathcal{D}'_k$,

$$v_\pm=\frac{1}{2\pi i}\int_\gamma(R_r^\pm(\omega,\lambda,k)\otimes R_\theta(\omega,\lambda))f\,d\lambda, \tag{3.14}$$

where

$$R_r^\pm(\omega,\lambda,k)=|r-r_\pm|^{iA_\pm^{-1}(1+\alpha)((r_\pm^2+a^2)\omega-ak)}R_r(\omega,\lambda,k).$$

By part 2 of Proposition 3.2, we may choose compact sets $K_\pm$ containing $r_\pm$ such that for each N, there exists a constant C_N (depending on ω, k, and γ) such that

$$\|1_{K_\pm}R_r^\pm(\omega,\lambda,k)1_{K_r}\|_{L^2\to C^N(K_\pm)}\leq\frac{C_N}{1+|\lambda|},\ \lambda\in\gamma.$$

(The estimate is true over a compact portion of γ since the image of $R_r^\pm$ consists of functions smooth at $r=r_\pm$, by the construction in Section 4.) Now, by (3.4) we get for some constant C_N',

$$\|R_r^\pm(\omega,\lambda,k)\otimes R_\theta(\omega,\lambda))f\|_{C^N(K_\pm;L^2(\mathbb{S}^2))}\leq\frac{C_N'\|f\|_{L^2}}{1+|\lambda|^2};$$

by (3.14), $v_\pm\in C^\infty(K_\pm;L^2(\mathbb{S}^2))$.

Now, since $(P_r+P_\theta)u=f$ and (assuming that $K_\pm\cap K_r=\emptyset$) $f|_{K_\pm\times\mathbb{S}^2}=0$, we have $(P_r^\pm(\omega,k)+P_\theta(\omega))v_\pm=0$ on $K_\pm\times\mathbb{S}^2$, where

$$P_r^\pm(\omega,k)=|r-r_\pm|^{iA_\pm^{-1}(1+\alpha)((r_\pm^2+a^2)\omega-ak)}P_r(\omega,k)|r-r_\pm|^{-iA_\pm^{-1}(1+\alpha)((r_\pm^2+a^2)\omega-ak)}$$

has smooth coefficients on $K_\pm$ (see Section 4). Then for each N,

$$P_\theta^N v_\pm=(-P_r^\pm)^N v_\pm\in C^\infty(K_\pm;L^2(\mathbb{S}^2));$$

since P_θ is elliptic, we get $v_\pm\in C^\infty(K_\pm;H^{2N}(\mathbb{S}^2))$. Therefore, $v_\pm\in C^\infty(K_\pm\times\mathbb{S}^2)$.

2. Let ω be a pole of $R_g(\omega,k)$. Then ω is not a regular point; therefore, there exists $\lambda\in\mathbb{C}$ such that (ω,λ) is a pole of both R_r and R_θ. This gives us functions $u_r(r)$ and $u_\theta(\theta,\varphi)\in\mathcal{D}_k'$ such that $(P_r(\omega,k)+\lambda)u_r=0$ and $(P_\theta(\omega)-\lambda)u_\theta=0$. It remains to take $u=u_r\otimes u_\theta$ and use part 4 of Proposition 3.2. □

The following fact will be used in the proof of Theorem 4, as well as in Section 7:

Proposition 3.4. *Fix $\delta_r>0$. Let ψ,C_r be the constants from part 2 of Proposition 3.2, C_θ,C_ψ be the constants from Proposition 3.1, and C_k be the constant from Proposition 3.3. Take $\omega\in\mathbb{C}$ and put*

$$L=(C_r(1+C_k)^2+C_\psi)(1+|\omega|)^2.$$

Assume that a is small enough so that Proposition 3.3 applies and suppose that ω and $l_1,l_2>0$ are chosen so that

$$l_1\geq C_\psi|a\omega|^2,\ l_2\geq C_\theta|a|(|a\omega|+C_k(1+|\omega|))|\operatorname{Im}\omega|,\ l_2\leq L\sin\psi. \tag{3.15}$$

Also, assume that for all λ and k satisfying

$$|k|\leq C_k(1+|\omega|),\ -l_1\leq\operatorname{Re}\lambda\leq L,\ |\operatorname{Im}\lambda|\leq l_2, \tag{3.16}$$

we have the estimate

$$\|1_{K_r}R_r(\omega,\lambda,k)1_{K_r}\|_{L^2\to L^2}\leq C_1 \tag{3.17}$$

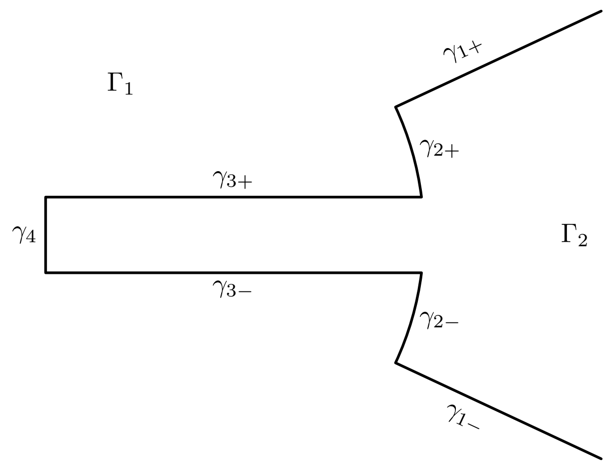

FIGURE 2. The admissible contour γ used in Proposition 3.4

for some constant C_1 independent of λ and k. Then ω is not a resonance and

$$\|R_g(\omega)\|_{L^2(M_K)\to L^2(M_K)} \leq C_2\left(\frac{1}{1+|\omega|^2}+\frac{1+C_1(l_1+1+|\omega|^2)}{l_2}+\frac{C_1 l_2}{l_1}\right) \tag{3.18}$$

for a certain global constant C_2.

Proof. First of all, by Proposition 3.3, it suffices to establish the estimate (3.18) for the operator $R_g(\omega,k)$, where $|k|\leq C_k(1+|\omega|)$. Now, by (2.1), it suffices to construct an admissible contour in the sense of Definition 2.3 and estimate the norms of R_r and R_θ on this contour. We take the contour γ composed of:

- the rays $\gamma_{1\pm}=\{\arg\lambda=\pm\psi, |\lambda|\geq L\}$;
- the arcs $\gamma_{2\pm}=\{|\arg\lambda|\leq\psi,\ |\lambda|=L,\ \pm\operatorname{Im}\lambda\geq l_2\}$;
- the segments $\gamma_{3\pm}$ of the lines $\{\operatorname{Im}\lambda=\pm l_2\}$ connecting $\gamma_{2\pm}$ with γ_4;
- the segment $\gamma_4=\{\operatorname{Re}\lambda=-l_1,\ |\operatorname{Im}\lambda|\leq l_2\}$.

Then γ divides the complex plane into two domains; we refer to the domain containing positive real numbers as Γ_2 and to the other domain as Γ_1. We claim that $R_\theta(\omega,\cdot)|_{\mathcal{D}'_k}$ has no poles in Γ_1, $R_r(\omega,\cdot,k)$ has no poles in Γ_2, and the $L^2\to L^2$ operator norm estimates

$$\|R_\theta(\omega,\lambda)\|\leq C/|\lambda|,\ \|1_{K_r}R_r(\omega,\lambda,k)1_{K_r}\|\leq C/|\lambda|,\ \lambda\in\gamma_{1\pm}; \tag{3.19}$$

$$\|R_\theta(\omega,\lambda)|_{\mathcal{D}'_k}\|\leq C/l_2,\ \|1_{K_r}R_r(\omega,\lambda,k)1_{K_r}\|\leq C/(1+|\omega|^2),\ \lambda\in\gamma_{2\pm}; \tag{3.20}$$

$$\|R_\theta(\omega,\lambda)|_{\mathcal{D}'_k}\|\leq C/l_2,\ \|1_{K_r}R_r(\omega,\lambda,k)1_{K_r}\|\leq C_1,\ \lambda\in\gamma_{3\pm}; \tag{3.21}$$

$$\|R_\theta(\omega,\lambda)\|\leq C/l_1,\ \|1_{K_r}R_r(\omega,\lambda,k)1_{K_r}\|\leq C_1,\ \lambda\in\gamma_4 \tag{3.22}$$

hold for some global constant C; then (3.18) follows from these estimates and (2.1).

First, we prove that $R_\theta(\omega,\cdot)|_{\mathcal{D}'_k}$ has no poles $\lambda \in \Gamma_1$. First of all, assume that $|\lambda| \geq L$. Then $|\arg\lambda| \geq \psi$ and we can apply part 3 of Proposition 3.1; we also get the first half of (3.19). Same argument works for $\operatorname{Re}\lambda \leq -l_1$, and we get the first half of (3.22). We may now assume that $|\lambda| \leq L$ and $\operatorname{Re}\lambda \geq -l_1$; it follows that $|\operatorname{Im}\lambda| \geq l_2$. But in that case, we can apply (3.3), and we get the first halves of (3.20) and (3.21).

Next, we prove that $R_r(\omega,\cdot,k)$ has no poles $\lambda \in \Gamma_2$. First of all, assume that $|\lambda| \geq L$ and $\operatorname{Re}\lambda \geq 0$. Then $|\arg\lambda| \leq \psi$ and we can apply part 2 of Proposition 3.2; we also get the second halves of (3.19) and (3.20). Now, in the opposite case, (3.16) is satisfied and we can use (3.17) to get the second halves of (3.21) and (3.22). ☐

Proof of Theorem 4. First, we take care of the resonances near zero. By Proposition 3.3, we can assume that k is bounded by some constant. Next, if $\omega = 0$ and $a = 0$, then $R_g(\omega,k)$ only has a pole for $k = 0$, and in the latter case, $\lambda = 0$ is the only common pole of $R_\theta(0,\cdot)$ and $R_r(0,\cdot,0)$. (In fact, the poles of $R_\theta(0,\cdot)|_{\mathcal{D}'_k}$ are given by $\lambda = l(l+1)$ for $l \geq |k|$; an integration by parts argument shows that $R_r(0,\cdot,k)$ cannot have poles with $\operatorname{Re}\lambda > 0$.) The sets of poles of the resolvents $R_\theta(\omega,\lambda)|_{\mathcal{D}'_k}$ and $R_r(\omega,\lambda,k)$ depend continuously on a in the sense that, if there are no poles of one of these resolvents for (ω,λ) in a fixed compact set for $a = 0$, then this is still true for a small enough. It follows from here and the first parts of Propositions 3.1 and 3.2 that there exists $\varepsilon_\omega,\varepsilon_\lambda > 0$ such that for a small enough,

- $R_g(\omega,k)$ does not have poles in $\{|\omega| \leq \varepsilon_\omega\}$ unless $k = 0$;
- if $|\omega| \leq \varepsilon_\omega$, then all common poles of $R_\theta(\omega,\cdot)|_{\mathcal{D}'_0}$ and $R_r(\omega,\cdot,0)$ lie in $\{|\lambda| \leq \varepsilon_\lambda\}$;
- the decompositions (3.1) and (3.6) hold for $|\omega| \leq \varepsilon_\omega$, $|\lambda| \leq \varepsilon_\lambda$;
- we have $\lambda_r(\omega) \neq \lambda_\theta(\omega)$ for $0 < |\omega| \leq \varepsilon_\omega$.

It follows immediately that $\omega = 0$ is the only pole of R_g in $\{|\omega| \leq \varepsilon_\omega\}$. To get the meromorphic decomposition, we repeat the argument at the end of Section 2 in our particular case. Note that for small $\omega \neq 0$,

$$R_g(\omega,0) = \frac{1}{2\pi i}\int_\gamma R_r(\omega,\lambda,0) \otimes R_\theta(\omega,\lambda)|_{\mathcal{D}'_0}\, d\lambda + \operatorname{Hol}(\omega)$$

Here γ is a small contour surrounding $\lambda_\theta(\omega)$, but not $\lambda_r(\omega)$; the integration is done in the clockwise direction; Hol denotes a family of operators holomorphic near zero. By (3.1)

and (3.6), we have

$$\begin{aligned}R_g(\omega,0) &= \mathrm{Hol}(\omega)+\frac{1}{2\pi i}\int_\gamma \frac{S_{r0}(\omega,\lambda)\otimes S_{\theta 0}(\omega,\lambda)}{(\lambda-\lambda_r(\omega))(\lambda-\lambda_\theta(\omega))}\,d\lambda\\ &= \mathrm{Hol}(\omega)+\frac{1}{\lambda_r(\omega)-\lambda_\theta(\omega)}\frac{1}{2\pi i}\int_\gamma (S_{r0}(\omega,\lambda)\otimes S_{\theta 0}(\omega,\lambda))\Big(\frac{1}{\lambda-\lambda_r(\omega)}-\frac{1}{\lambda-\lambda_\theta(\omega)}\Big)\,d\lambda\\ &= \mathrm{Hol}(\omega)+\frac{1}{\lambda_r(\omega)-\lambda_\theta(\omega)}S_{r0}(\omega,\lambda_\theta(\omega))\otimes S_{\theta 0}(\omega,\lambda_\theta(\omega))\\ &= \mathrm{Hol}(\omega)+\frac{i(1\otimes 1)}{4\pi(1+\alpha)(r_+^2+r_-^2+2a^2)\omega}.\end{aligned}$$

Now, let us consider the case $|\omega|>\varepsilon_\omega$, $\operatorname{Im}\omega>0$. We will apply Proposition 3.4 with $l_1=|\omega|^2/C_{1r}$, $l_2=|\omega|\operatorname{Im}\omega/C_{1r}$. Here C_{1r} is the constant in Proposition 3.2. Then (3.15) is true for small a and (3.17) follows from (3.16) for small a by part 5 of Proposition 3.2, with $C_1=C_{1r}/(|\omega|\operatorname{Im}\omega)$. It remains to use (3.18).

Finally, assume that ω is a real k-resonance and $|\omega|>\varepsilon_\omega$. Then by Proposition 3.3, and part 3 of Proposition 3.2, if a is small enough, then the operator $R_r(\omega,\cdot,k)$ cannot have a pole for $\lambda\in\mathbb{R}$. However, the operator $P_\theta(\omega)$ is self-adjoint and thus only has real eigenvalues, a contradiction. □

4. Construction of the radial resolvent

In this section, we prove Proposition 3.2, except for part 2, which is proved in Section 6. We start with a change of variables that maps (r_-,r_+) to $(-\infty,\infty)$:

Proposition 4.1. *Define $x=x(r)$ by*

$$x=\int_{r_0}^r \frac{ds}{\Delta_r(s)}. \tag{4.1}$$

(Here $r_0\in(r_-,r_+)$ is a fixed number.) Then there exists a constant X_0 such that for $\pm x>X_0$, we have $r=r_\pm\mp F_\pm(e^{\mp A_\pm x})$, where $F_\pm(w)$ are real analytic on $[0,e^{-A_\pm X_0})$ and holomorphic in the discs $\{|w|<e^{-A_\pm X_0}\}\subset\mathbb{C}$.

Proof. We concentrate on the behavior of x near r_+. It is easy to see that $-A_+x(r)=\ln(r_+-r)+G(r)$, where G is holomorphic near $r=r_+$. Exponentiating, we get

$$w=e^{-A_+x}=(r_+-r)e^{G(r)}.$$

It remains to apply the inverse function theorem to solve for r as a function of w near zero. □

After the change of variables $r \to x$, we get $P_r(\omega, k) + \lambda = \Delta_r^{-1} P_x(\omega, \lambda, k)$, where

$$\begin{gathered} P_x(\omega, \lambda, k) = D_x^2 + V_x(x; \omega, \lambda, k), \\ V_x = \lambda \Delta_r - (1+\alpha)^2((r^2+a^2)\omega - ak)^2. \end{gathered} \tag{4.2}$$

(We treat r and Δ_r as functions of x now.) We put

$$\omega_\pm = (1+\alpha)((r_\pm^2 + a^2)\omega - ak), \tag{4.3}$$

so that $V_x(\pm\infty) = -\omega_\pm^2$. Also, by Proposition 4.1, we get

$$V_x(x) = V_\pm(e^{\mp A_\pm x}), \ \pm x > X_0, \tag{4.4}$$

where $V_\pm(w)$ are functions holomorphic in the discs $\{|w| < e^{-A_\pm X_0}\}$.

We now define outgoing functions:

Definition 4.1. *Fix ω, k, λ. A function $u(x)$ (and the corresponding function of r) is called outgoing at $\pm\infty$ iff*

$$u(x) = e^{\pm i\omega_\pm x} v_\pm(e^{\mp A_\pm x}), \tag{4.5}$$

where $v_\pm(w)$ are holomorphic in a neighborhood of zero. We call $u(x)$ outgoing if it is outgoing at both infinities.

Let us construct certain solutions outgoing at one of the infinities:

Proposition 4.2. *There exist solutions $u_\pm(x; \omega, \lambda, k)$ to the equation $P_x u_\pm = 0$ of the form*

$$u_\pm(x; \omega, \lambda, k) = e^{\pm i\omega_\pm x} v_\pm(e^{\mp A_\pm x}; \omega, \lambda, k),$$

where $v_\pm(w; \omega, \lambda, k)$ is holomorphic in $\{|w| < W_\pm\}$ and

$$v_\pm(0; \omega, \lambda, k) = \frac{1}{\Gamma(1 - 2i\omega_\pm A_\pm^{-1})}. \tag{4.6}$$

These solutions are holomorphic in (ω, λ) and are unique unless $\nu = 2i\omega_\pm A_\pm^{-1}$ is a positive integer.

Proof. We only construct the function u_+. Let us write the Taylor series for v_+ at zero:

$$v_+(w) = \sum_{j \geq 0} v_j w^j.$$

Put $w = e^{-A_+ x}$; then the equation $P_x u_+ = 0$ is equivalent to

$$((A_+ w D_w - \omega_+)^2 + V_x) v_\pm = 0.$$

By (4.2) and Proposition 4.1, V_x is a holomorphic function of w for $|w| < W_+$. If $V_x = \sum_{j \geq 0} V_j w^j$ is the corresponding Taylor series, then we get the following system of linear equations on the coefficients v_j:

$$j A_+(2i\omega_+ - j A_+) v_j + \sum_{0 < l \leq j} V_l v_{j-l} = 0, \ j > 0. \tag{4.7}$$

If ν is not a positive integer, then this system has a unique solution under the condition $v_0 = \Gamma(1-\nu)^{-1}$. This solution can be uniquely holomorphically continued to include the cases when ν is a positive integer. Indeed, one defines the coefficients $v_0, \dots, v_\nu$ by Cramer's Rule using the first ν equations in (4.7) (this can be done since the zeroes of the determinant of the corresponding matrix match the poles of the gamma function), and the rest are uniquely determined by the remaining equations in the system (4.7).

We now prove that the series above converges in the disc $\{|w| < W_+\}$. We take $\varepsilon > 0$; then $|V_j| \leq M(W_+ - \varepsilon)^{-j}$ for some constant M. Then one can use induction and (4.7) to see that $|v_j| \leq C(W_+ - \varepsilon)^{-j}$ for some constant C. Therefore, the Taylor series for v converges in the disc $\{|w| < W_+ - \varepsilon\}$; since ε was arbitrary, we are done. □

The condition (4.6) makes it possible for $u_\pm$ to be zero for certain values of $\omega_\pm$. However, we have the following

Proposition 4.3. *Assume that one of the solutions $u_\pm$ is identically zero. Then every solution u to the equation $P_x u = 0$ is outgoing at the corresponding infinity.*

Proof. Assume that $u_+(x; \omega_0, \lambda_0, k_0) \equiv 0$. (The argument for u_- is similar.) Put $\nu = 2i\omega_{0+}A_+^{-1}$; by (4.6), it has to be a positive integer. Similarly to Proposition 4.2, we can construct a nonzero solution u_1 to the equation $P_x u_1 = 0$ with

$$u_1(x) = e^{-i\omega_{0+}x}\tilde{v}_1(e^{-A_+x})$$

and $\tilde{v}_1$ holomorphic at zero. We can see that $u_1(x) = e^{i\omega_{0+}x}v_1(e^{-A_+x})$, where $v_1(w) = w^\nu \tilde{v}_1(w)$ is holomorphic; therefore, u_1 is outgoing. Note that $u_1(x) = o(e^{i\omega_{0+}x})$ as $x \to +\infty$.

Now, since $u_+(x; \omega_0, \lambda_0, k_0) \equiv 0$, we can define

$$u_2(x) = \lim_{\omega \to \omega_0} \Gamma(1 - 2i\omega_+ A_+^{-1}) u_+(x; \omega, \lambda_0, k_0);$$

it will be an outgoing solution to the equation $P_x u_2 = 0$ and have $u_2(x) = e^{i\omega_{0+}x}(1 + o(1))$ as $x \to +\infty$. We have constructed two linearly independent outgoing solutions to the equation $P_x u = 0$; since this equation only has a two-dimensional space of solutions, every its solution must be outgoing. □

The next statement follows directly from the definition of an outgoing solution and will be used in later sections:

Proposition 4.4. *Fix $\delta_r > 0$ and let K_x be the image of the set $K_r = (r_- + \delta_r, r_+ - \delta_r)$ under the change of variables $r \to x$. Assume that X_0 is chosen large enough so that Proposition 4.1 holds and $K_x \subset (-X_0, X_0)$. Let $u(x) \in H^2_{\mathrm{loc}}(\mathbb{R})$ be any outgoing function in the sense of Definition 4.1 and assume that $f = P_x u$ is supported in K_x. Then:*

1. u can be extended holomorphically to the two half-planes $\{\pm \operatorname{Re} z > X_0\}$ and satisfies the equation $P_z u = 0$ in these half-planes, where $P_z = D_z^2 + V_x(z)$ and $V_x(z)$ is well-defined by (4.4).

2. If γ is a contour in the complex plane given by $\operatorname{Im} z = F(\operatorname{Re} z)$, $x_- \leq \operatorname{Re} z \leq x_+$, and $F(x) = 0$ for $|x| \leq X_0$, then we can define the restriction to γ of the holomorphic extension of u by

$$u_\gamma(x) = u(x + iF(x))$$

and u_γ satisfies the equation $P_\gamma u_\gamma = f$, where

$$P_\gamma = \left(\frac{1}{1 + iF'(x)} D_x\right)^2 + V_x(x + iF(x)).$$

3. Assume that γ is as above, with $x_\pm = \pm\infty$, and $F'(x) = c = \mathrm{const}$ for large $|x|$. Then $u_\gamma(x) = O(e^{\mp \operatorname{Im}((1+ic)\omega_\pm)x})$ as $x \to \pm\infty$. As a consequence, if $\operatorname{Im}((1 + ic)\omega_\pm) > 0$, then $u_\gamma(x) \in H^2(\mathbb{R})$.

We are now ready to prove Proposition 3.2.

Proof of part 1. Given the functions $u_\pm$, define the operator $S_x(\omega, \lambda, k)$ on $\mathbb{R}$ by its Schwartz kernel

$$S_x(x, x'; \omega, \lambda, k) = u_+(x)u_-(x')[x > x'] + u_-(x)u_+(x')[x < x'].$$

The operator $S_x(\omega, \lambda)$ acts $L^2_{\mathrm{comp}}(\mathbb{R}) \to H^2_{\mathrm{loc}}(\mathbb{R})$ and $P_x S_x = W(\omega, \lambda, k)$, where the Wronskian

$$W(\omega, \lambda, k) = u_+(x; \omega, \lambda, k) \cdot \partial_x u_-(x; \omega, \lambda, k) - u_-(x; \omega, \lambda, k) \cdot \partial_x u_+(x; \omega, \lambda, k)$$

is constant in x. Moreover, $W(\omega, \lambda, k) = 0$ if and only if $u_+(x; \omega, \lambda, k)$ and $u_-(x; \omega, \lambda, k)$ are linearly dependent as functions of x. Also, the image of S_x consists of outgoing functions.

Now, we define the radial resolvent $R_r(\omega, \lambda, k) = R_x(\omega, \lambda, k)\Delta_r$, where

$$R_x(\omega, \lambda, k) = \frac{S_x(\omega, \lambda, k)}{W(\omega, \lambda, k)}. \tag{4.8}$$

It is clear that R_r is a meromorphic family of operators $L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$ and $(P_r + \lambda)R_r$ is the identity operator. We now prove that R_x, and thus R_r, has poles of finite rank. Fix k and take $(\omega_0, \lambda_0) \in \{W = 0\}$; we need to prove that for every l, the principal part of the Laurent decomposition of $\partial^l_\omega R_x(\omega_0, \lambda, k)$ at $\lambda = \lambda_0$ consists of finite-dimensional operators. We use induction on l. One has $P_x(\omega, \lambda, k)R_x(\omega, \lambda, k) = 1$; differentiating this identity l times in ω, we get

$$P_x(\omega_0, \lambda, k)\partial^l_\omega R_x(\omega_0, \lambda, k) = \delta_{l0}1 + \sum_{m=1}^{l} c_{ml}\partial^m_\omega P_x(\omega_0, \lambda, k)\partial^{l-m}_\omega R_x(\omega_0, \lambda, k).$$

(Here c_{ml} are some constants.) The right-hand side has poles of finite rank by the induction hypothesis. Now, consider the Laurent decomposition

$$\partial^l_\omega R_x(\omega_0, \lambda, k) = Q(\lambda) + \sum_{j=1}^{N} \frac{R_j}{(\lambda - \lambda_0)^j}.$$

Here Q is holomorphic at λ_0. Multiplying by P_x, we get

$$\sum_{j=1}^{N}\frac{P_x(\omega_0,\lambda,k)R_j}{(\lambda-\lambda_0)^j}\sim\sum_{j=1}^{N}\frac{L_j}{(\lambda-\lambda_0)^j}$$

up to operators holomorphic at λ_0. Here L_j are some finite-dimensional operators. We then have

$$P_x(\omega_0,\lambda_0,k)R_N=L_N,$$
$$P_x(\omega_0,\lambda_0,k)R_{N-1}=L_{N-1}-(\partial_\lambda P_x(\omega_0,\lambda_0,k))R_N,\ \dots$$

Each of the right-hand sides has finite rank and the kernel of $P_x(\omega_0,\lambda_0,k)$ is two-dimensional; therefore, each R_j is finite-dimensional as required. (We also see immediately that the image of each R_j consists of smooth functions.)

Finally, we establish the decomposition at zero. As in part 1 of Proposition 3.1, it suffices to compute $S_x(0,0,0)$ and the first order terms in the Taylor expansion of W at $(0,0,0)$. We have $u_\pm(x;0,0,0)=1$ for all x; therefore, $S_x(x,x';0,0,0)=1$. Next, put $u_{\omega\pm}(x)=\partial_\omega u_\pm(x;0,0,0)$ and $u_{\lambda\pm}(x)=\partial_\lambda u_\pm(x;0,0,0)$. By differentiating the equation $P_xu_\pm=0$ in ω and λ and recalling the boundary conditions at $\pm\infty$, we get

$$\partial_x^2u_{\lambda\pm}(x)=\Delta_r,$$
$$u_{\lambda\pm}(x)=v_{\lambda\pm}(e^{\mp A_\pm x}),\ \pm x\gg 0;$$
$$\partial_x^2u_{\omega\pm}(x)=0,$$
$$u_{\omega\pm}(x)=\pm i(1+\alpha)(r_\pm^2+a^2)x+v_{\omega\pm}(e^{\mp A_\pm x}),\ \pm x\gg 0,$$

for some functions $v_{\lambda\pm},v_{\omega\pm}$ real analytic at zero. We then find

$$\partial_\lambda W(0,0,0)=\partial_x(u_{-\lambda}-u_{+\lambda})=\int_{-\infty}^{\infty}\Delta_r\,dx=r_+-r_-,$$
$$\partial_\omega W(0,0,0)=\partial_x(u_{-\omega}-u_{+\omega})=-i(1+\alpha)(r_+^2+r_-^2+2a^2).\quad\square$$

Proof of part 3. Assume that ω and λ are both real and R_r has a pole at (ω,λ,k). Let $u(x)$ be the corresponding resonant state; we know that it has the asymptotics

$$u_\pm(x)=e^{\pm i\omega_\pm x}U_\pm(1+O(e^{\mp A_\pm x})),\ x\to\pm\infty;$$
$$\partial_xu_\pm(x)=e^{\pm i\omega_\pm x}U_\pm(\pm i\omega_\pm+O(e^{\mp A_\pm x})),\ x\to\pm\infty$$

for some nonzero constants $U_\pm$. Since $V_x(x;\omega,\lambda,k)$ is real-valued, both u and $\bar u$ solve the equation $(D_x^2+V_x(x))u=0$. Then the Wronskian $W_u(x)=u\cdot\partial_x\bar u-\bar u\cdot\partial_xu$ must be constant; however,

$$W_u(x)\to\mp 2i\omega_\pm|U_\pm|^2\text{ as }x\to\pm\infty.$$

Then we must have $\omega_+\omega_-\le 0$; it follows immediately that $|\omega|=O(|ak|)$. $\square$

Proof of part 4. First, assume that neither of $u_\pm$ is identically zero. Then the resolvent R_x, and thus R_r, has a pole iff the functions $u_\pm$ are linearly dependent, or, in other words, if there exists a nonzero outgoing solution $u(x)$ to the equation $P_x u = 0$. Now, if one of $u_\pm$, say, u_+, is identically zero, then by Proposition 4.3, u_- will be an outgoing solution at both infinities. □

Proof of part 5. Assume that $u(x)$ is outgoing and $P_x(\omega,\lambda,k)u = f \in L^2(K_x)$. Since $\operatorname{Im}\omega > 0$, we have $\operatorname{Im}\omega_\pm > 0$ and thus $u \in H^2(\mathbb{R})$.

First, assume that $|\arg\omega - \pi/2| < \varepsilon$, where $\varepsilon > 0$ is a constant to be chosen later. Then

$$\operatorname{Re} V_x(x) = (1+\alpha)^2(r^2+a^2)^2(\operatorname{Im}\omega)^2 + \operatorname{Re}\lambda\cdot\Delta_r - (1+\alpha)^2((r^2+a^2)\operatorname{Re}\omega - ak)^2;$$

using (3.10), we can choose ε and C_{1r} so that $\operatorname{Re} V_x(x) \geq |\omega|^2/C > 0$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned}\|u\|_{L^2(\mathbb{R})}\cdot\|f\|_{L^2(\mathbb{R})} &\geq \operatorname{Re}\int \bar{u}(x)(D_x^2 + V_x(x))u(x)\,dx\\ &\geq \int \operatorname{Re} V_x(x)|u|^2\,dx \geq C^{-1}|\omega|^2\|u\|^2_{L^2(\mathbb{R})}\end{aligned}$$

and (3.11) follows.

Now, assume that $|\arg\omega - \pi/2| \geq \varepsilon$. Then

$$\operatorname{Im} V_x(x) = -2(1+\alpha)^2((r^2+a^2)\operatorname{Re}\omega - ak)(r^2+a^2)\operatorname{Im}\omega + \operatorname{Im}\lambda\cdot\Delta_r;$$

it follows from (3.10) that we can choose C_{1r} so that the sign of $\operatorname{Im} V_x(x)$ is constant in x (positive if $\arg\omega > \pi/2$ and negative otherwise) and, in fact, $|\operatorname{Im} V_x(x)| \geq |\omega|\operatorname{Im}\omega/C > 0$ for all x. Then (assuming that $\operatorname{Im} V_x(x) > 0$)

$$\begin{aligned}\|u\|_{L^2(\mathbb{R})}\cdot\|f\|_{L^2(\mathbb{R})} &\geq \operatorname{Im}\int \bar{u}(x)(D_x^2 + V_x(x))u(x)\,dx\\ &= \int \operatorname{Im} V_x(x)|u|^2\,dx \geq C^{-1}|\omega|\operatorname{Im}\omega\|u\|^2_{L^2(\mathbb{R})}\end{aligned}$$

and (3.11) follows. □

5. Preliminaries from semiclassical analysis

In this section, we list certain facts from semiclassical analysis needed in the further analysis of our radial operator. For a general introduction to semiclassical analysis, the reader is referred to [17].

Let $a(x,\xi)$ belong to the symbol class

$$S^m = \{a(x,\xi) \in C^\infty(\mathbb{R}^2) \mid \sup_{x,\xi}\langle\xi\rangle^{|\beta|-m}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \text{ for all } \alpha,\beta\}.$$

Here $m \in \mathbb{R}$ and $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. Following [17, Section 8.6], we define the corresponding semiclassical pseudodifferential operator $a^w(x, hD_x)$ by the formula

$$a^w(x, hD_x)u(x) = \frac{1}{2\pi h}\int e^{\frac{i}{h}(x-y)\eta} a\Big(\frac{x+y}{2}, \eta\Big) u(y)\, dy d\eta.$$

Here $h > 0$ is the semiclassical parameter. We denote by Ψ^m the class of all semiclassical pseudodifferential operators with symbols in S^m. Introduce the semiclassical Sobolev spaces $H_h^l \subset \mathcal{D}'(\mathbb{R})$ with the norm $\|u\|_{H_h^l} = \|\langle hD_x\rangle^l u\|_{L^2}$; then for $a \in S^m$, we have

$$\|a^w(x, hD_x)\|_{H_h^l \to H_h^{l-m}} \leq C,$$

where C is a constant depending on a, but not on h. Also, if $a(x,\xi) \in C_0^\infty(\mathbb{R}^2)$, then

$$\|a^w(x, hD_x)\|_{L^2(\mathbb{R}) \to L^\infty(\mathbb{R})} \leq Ch^{-1/2}, \tag{5.1}$$

where C is a constant depending on a, but not on h. (See [17, Theorem 7.10] for the proof.)

General facts on multiplication of pseudodifferential operators can be found in [17, Section 8.6]. We will need the following: for $a \in S^m$ and $b \in S^n$, [5]

$$\text{if } \operatorname{supp} a \cap \operatorname{supp} b = \emptyset, \text{ then } a^w(x, hD_x) b^w(x, hD_x) = O_{L^2 \to H_h^N}(h^\infty) \text{ for all } N; \tag{5.2}$$

$$a^w(x, hD_x) b^w(x, hD_x) = (ab)^w(x, hD_x) + O_{\Psi^{m+n-1}}(h), \tag{5.3}$$

$$[a^w(x, hD_x), b^w(x, hD_x)] = -ih\{a, b\}^w(x, hD_x) + O_{\Psi^{m+n-2}}(h^2). \tag{5.4}$$

Here $\{\cdot,\cdot\}$ is the Poisson bracket, defined by $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_\xi b \cdot \partial_x a$. Also, if $A \in \Psi^m$, then the adjoint operator A^* also lies in Ψ^m and its symbol is the complex conjugate of the symbol of A.

One can study pseudodifferential operators on manifolds [17, Appendix E], and on particular on the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. If $a(x,\xi) = a(\xi)$ is a symbol on $T^*\mathbb{S}^1$ that is independent of x, then $a^w(hD_x)$ is a Fourier series multiplier modulo $O(h^\infty)$: for each N,

$$\text{if } u(x) = \sum_{j\in\mathbb{Z}} u_j e^{ijx}, \text{ then } a(hD_x)u(x) = \sum_{j\in\mathbb{Z}} a(hj) u_j e^{ijx} + O_{H_h^N}(h^\infty)\|u\|_{L^2}. \tag{5.5}$$

In the next three propositions, we assume that $P(h) \in \Psi^m$ and $P(h) = p^w(x, hD_x) + O_{\Psi^{m-1}}(h)$, where $p(x,\xi) \in S^m$.

Proposition 5.1. *(Elliptic estimate) Suppose that the function $\chi \in S^0$ is chosen so that $|p| \geq \langle\xi\rangle^m / C > 0$ on $\operatorname{supp}\chi$ for some h-independent constant C. Also, assume that either the set $\operatorname{supp}\chi$ or its complement is precompact. Then there exists a constant C_1 such that for each $u \in H_h^m$,*

$$\|\chi^w(x, hD_x)u\|_{H_h^m} \leq C_1\|P(h)u\|_{L^2} + O(h^\infty)\|u\|_{L^2}. \tag{5.6}$$

[5] We write $A(h) = O_X(h^k)$ for some Fréchet space X, if for each seminorm $\|\cdot\|_X$ of X, there exists a constant C such that $\|A(h)\|_X \leq Ch^k$. We write $A(h) = O_X(h^\infty)$ if $A(h) = O_X(h^k)$ for all k.

Proof. The proof follows the standard parametrix construction. We find a sequence of symbols $q_j(x,\xi;h) \in S^{-m-j}$, $j \geq 0$, such that for

$$Q_N(h) = \sum_{0 \leq j \leq N} h^j q_j^w(x, hD_x),$$

we get

$$(Q_N(h)P(h) - 1)\chi^w(x, hD_x) = O_{\Psi^{-N-1}}(h^{N+1}); \tag{5.7}$$

applying this operator equation to u, we prove the proposition.

We can take any $q_0 \in \Psi^{-m}$ such that $q_0 = p^{-1}$ near $\operatorname{supp}\chi$; such a symbol exists under our assumptions. The rest of q_j can be constructed by induction using the equation (5.7). □

Proposition 5.2. *(Gårding inequalities) Suppose that $\chi \in C_0^\infty(\mathbb{R}^2)$.*

1. If $\operatorname{Re} p \geq 0$ near $\operatorname{supp}\chi$, then there exists a constant C such that for every $u \in L^2$,

$$\operatorname{Re}(P(h)\chi^w u, \chi^w u) \geq -Ch\|\chi^w u\|_{L^2}^2 - O(h^\infty)\|u\|_{L^2}^2. \tag{5.8}$$

2. If $\operatorname{Re} p \geq 2\varepsilon > 0$ near $\operatorname{supp}\chi$ for some constant $\varepsilon > 0$, then for h small enough and every $u \in L^2$,

$$\operatorname{Re}(P(h)\chi^w u, \chi^w u) \geq \varepsilon\|\chi^w u\|_{L^2}^2 - O(h^\infty)\|u\|_{L^2}^2. \tag{5.9}$$

Proof. 1. Take $\chi_1 \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ such that $\chi_1 = 1$ near $\operatorname{supp}\chi$, but $\operatorname{Re} p \geq 0$ near $\operatorname{supp}\chi_1$. Then, apply the standard sharp Gårding inequality [17, Theorem 4.24] to the operator $\chi_1^w P(h)\chi_1^w$ and the function $\chi^w u$, and use (5.2).

2. Apply part 1 of this proposition to the operator $P(h) - 2\varepsilon$. □

Proposition 5.3. *(Exponentiation of pseudodifferential operators) Assume that $G \in C_0^\infty(\mathbb{R}^2)$, $s \in \mathbb{R}$, and define the operator $e^{sG^w} : L^2 \to L^2$ as*

$$e^{sG^w} = \sum_{j \geq 0} \frac{(sG^w)^j}{j!}.$$

Assume that $|s|$ is bounded by an h-independent constant. Then:

1. $e^{sG^w} \in \Psi^0$ is a pseudodifferential operator.

2. $e^{sG^w} P(h) e^{-sG^w} = P(h) + ish(H_p G)^w + O_{L^2 \to L^2}(h^2)$.

Proof. 1. See for example [17, Theorem 8.3] (with $m(x,\xi) = 1$). The full symbol of e^{sG^w} can be recovered from the evolution equation satisfied by this family of operators; we see that it is equal to 1 outside of a compact set.

2. It suffices to differentiate both sides of the equation in s, divide them by h, and compare the principal symbols. □

6. Analysis near the zero energy

In this section, we prove part 2 of Proposition 3.2. Take $h > 0$ such that $\operatorname{Re}\lambda = h^{-2}$. Put

$$\tilde\mu = h^2 \operatorname{Im}\lambda,\ \tilde k = hk,\ \tilde\omega = h\omega,\ \tilde\omega_\pm = h\omega_\pm;$$

then (3.7) implies that

$$|\mu| \leq \varepsilon_r,\ |a\tilde k| \leq \varepsilon_r,\ |\tilde\omega| \leq \varepsilon_r,\ |\tilde\omega_\pm| \leq \varepsilon_r, \tag{6.1}$$

where $\varepsilon_r > 0$ and h can be made arbitrarily small by choice of C_r and ψ. If P_x is the operator in (4.2), then $P_x = h^{-2}\tilde P_x$, where

$$\tilde P_x(h;\tilde\omega,\tilde\mu,\tilde k) = h^2 D_x^2 + \widetilde V_x(x;\tilde\omega,\tilde\mu,\tilde k),$$
$$\widetilde V_x(x;\tilde\omega,\tilde\mu,\tilde k) = (1+i\tilde\mu)\Delta_r - (1+\alpha)^2((r^2+a^2)\tilde\omega - a\tilde k)^2.$$

Now, we use Proposition 4.4. Let u be an outgoing function in the sense of Definition 4.1 and assume that $f = \tilde P_x u$ is supported in K_x. Then u satisfies (4.5) for $|x| > X_0$ and some functions $v_\pm$. Fix $x_+ > X_0$ and consider the function

$$v_1(y) = v_+(e^{-A_+(x_+ + iy)};\omega,\lambda,k),\ y \in \mathbb{R}. \tag{6.2}$$

This is a $2\pi/A_+$-periodic function; we can think of it as a function on the circle. It follows from the differential equation satisfied by v_+ together with Cauchy-Riemann equations that $Q(h)v_1(y) = 0$, where

$$Q(h;\tilde\omega,\tilde\mu,\tilde k) = (-ihD_y + \tilde\omega_+)^2 + \widetilde V_x(x_+ + iy;\tilde\omega,\tilde\mu,\tilde k).$$

Let $q(y,\eta)$ be the semiclassical symbol of Q:

$$q(y,\eta) = (-i\eta + \tilde\omega_+)^2 + \widetilde V_x(x_+ + iy).$$

For small h, the function $v_1(y)$ has to be (semiclassically) microlocalized on the set $\{q = 0\}$. Since the symbol q is complex-valued, in a generic situation this set will consist of isolated points. Also, since v_1 is the restriction to a certain circle of the function v_+, which is holomorphic inside this circle, it is microlocalized in $\{\eta \leq 0\}$. Therefore, if the equation $q(y,\eta) = 0$ has only one root with $\eta \leq 0$, then the function v_1 has to be microlocalized at this root. If furthermore $\bar q$ satisfies Hörmander's hypoellipticity condition, one can obtain an asymptotic decomposition of v_1 in powers of h. We will only need a weak corollary of such decomposition; here is a self-contained proof of the required estimates:

Proposition 6.1. *Assume that $x_+ > X_0$ is chosen so that:*

- *the equation $q(y,\eta) = 0$, $y \in \mathbb{S}^1$, has exactly one root (y_0,η_0) such that $\eta_0 < 0$;*
- *the equation $q(y,\eta) = 0$ has no roots with $\eta = 0$;*
- *the condition $i\{q,\bar q\} < 0$ is satisfied at (y_0,η_0);*
- $\operatorname{Re}(\eta_0 + i\tilde\omega_+) < 0$.

*(If all of the above hold, we say that we have **vertical control** at x_+ and (y_0, η_0) is called the **microlocalization point**.) Let $\eta(y)$ be the family of solutions to $q(y, \eta(y)) = 0$ with $\eta(y_0) = \eta_0$. Then for each N, each $\chi(y, \eta) \in C_0^\infty$ that is equal to 1 near (y_0, η_0), and h small enough, we have*

$$\|(1 - \chi^w(y, hD_y))v_1\|_{H_h^N} = O(h^\infty)\|v_1\|_{L^2}, \tag{6.3}$$

$$\|(hD_y - \eta(y))v_1\|_{H_h^N} = O(h)\|v_1\|_{L^2}, \tag{6.4}$$

$$\|v_1\|_{L^2} \leq Ch^{1/4}|v_1(y_0)|, \tag{6.5}$$

$$|(hD_y - \eta_0)v_1(y_0)| \leq Ch^{1/2}\|v_1\|_{L^2}, \tag{6.6}$$

$$\mathrm{Re}\left(\frac{h\partial_x u_+(x_+ + iy_0)}{u_+(x_+ + iy_0)}\right) \leq -\frac{1}{C} < 0. \tag{6.7}$$

Similar statements are true for u_+ replaced by u_-, with the opposite inequality sign in (6.7).

Proof. (6.3)*:* We know that

$$\inf\{\eta \mid q(y, \eta) = 0,\ (y, \eta) \neq (y_0, \eta_0)\} > 0.$$

Therefore, we can decompose $1 = \chi + \chi_+ + \chi_0$, where χ_+ depends only on the η variable, is supported in $\{\eta > 0\}$, and is equal to 1 for large positive η and near every root of the equation $q(y, \eta) = 0$ with $\eta > 0$. Since v_+ is holomorphic at zero, its Taylor series provides the Fourier series for v_1; it then follows from (5.5) that

$$\|\chi_+^w(y, hD_y)v_1\|_{H_h^N} = O(h^\infty)\|v_1\|_{L^2}.$$

Next, the symbol q is elliptic near $\operatorname{supp}\chi_0$; therefore, by Proposition 5.1 (whose proof applies without changes to our case), since $Q(h)v_1 = 0$, we have

$$\|\chi_0^w(y, hD_y)v_1\|_{H_h^N} = O(h^\infty)\|v_1\|_{L^2}.$$

This finishes the proof.

(6.4)*:* Take a small cutoff χ as above, and factor $q = (\eta - \eta(y))q_1$, where $q_1(y, \eta)$ is nonzero near $\operatorname{supp}\chi$. We then find a compactly supported symbol r_1 with $r_1 q_1 = 1$ near $\operatorname{supp}\chi$. Now, we have

$$\|\chi^w(y, hD_y)(r_1^w(y, hD_y)q_1^w(y, hD_y) - 1)(hD_y - \eta(y))v_1\|_{H_h^N} = O(h)\|v_1\|_{L^2},$$

$$\|(1 - \chi^w(y, hD_y))(r_1^w(y, hD_y)q_1^w(y, hD_y) - 1)(hD_y - \eta(y))v_1\|_{H_h^N} = O(h^\infty)\|v_1\|_{L^2},$$

$$\|r_1^w(y, hD_y)(q_1^w(y, hD_y)(hD_y - \eta(y)) - Q(h))v_1\|_{H_h^N} = O(h)\|v_1\|_{L^2}.$$

It remains to add these up.

(6.5)*:* We cut off v_1 to make it supported in a small ε-neighborhood of y_0. Put $f = (h\partial_y - i\eta(y))v_1$; we know that $\|f\|_{L^2} \leq Ch\|v_1\|_{L^2}$. Now, put

$$\Phi(y) = \int_{y_0}^{y} \eta(y')\, dy'.$$

The condition $i\{q,\bar q\}|_{(y_0,\eta_0)}<0$ is equivalent to

$$\operatorname{Im}\partial_y\eta(y_0)>0;$$

it follows that

$$\operatorname{Im}(\Phi(y)-\Phi(y'))\geq\beta((y-y_0)^2-(y'-y_0)^2) \tag{6.8}$$

for some $\beta>0$, $|y-y_0|<\varepsilon$, and y' between y and y_0. (To see that, represent the left-hand side as an integral.) Now,

$$v_1(y)=e^{i\Phi(y)/h}v_1(y_0)+h^{-1}\int_{y_0}^{y}e^{i(\Phi(y)-\Phi(y'))/h}f(y')\,dy'.$$

Let $Tf(y)$ be the second term in the sum above; it suffices to prove that

$$\|Tf\|_{L^2(y_0-\varepsilon,y_0+\varepsilon)}\leq Ch^{-1/2}\|f\|_{L^2(y_0-\varepsilon,y_0+\varepsilon)}.$$

This can be reduced to the inequalities

$$\sup_{0\leq y-y_0<\varepsilon}\int_{y_0}^{y}|e^{i(\Phi(y)-\Phi(y'))/h}|\,dy'=O(h^{1/2}),$$

$$\sup_{0\leq y'-y_0<\varepsilon}\int_{y'}^{y_0+\varepsilon}|e^{i(\Phi(y)-\Phi(y'))/h}|\,dy=O(h^{1/2}).$$

and similar inequalities for the case $y,y'<y_0$. We now use (6.8); after a change of variables, it suffices to prove that

$$\sup_{y>0}\int_0^y e^{(y')^2-y^2}\,dy'<\infty,\ \sup_{y'>0}\int_{y'}^{\infty}e^{(y')^2-y^2}\,dy<\infty.$$

To prove the first of these inequalities, make the change of variables $y'=ys$; then the integral becomes

$$\int_0^1 ye^{y^2(s^2-1)}\,ds.$$

However, $ye^{y^2(s^2-1)}\leq C(1-s^2)^{-1/2}$, and the integral of the latter converges.

After the change of variables $y=y'+s$, the integral of the second inequality above becomes

$$\int_0^{\infty}e^{-2y's-s^2}\,ds.$$

This can be estimated by $\int e^{-s^2}\,ds$.

(6.6)*:* Let $\chi\in C_0^\infty(\mathbb{R}^2)$ have $\chi=1$ near (y_0,η_0). Combining (5.1) and (6.4) with

$$\|(1-\chi^w(y,hD_y))(hD_y-\eta(y))v_1\|_{L^\infty}=O(h^\infty)\|v_1\|_{L^2},$$

we get $\|(hD_y-\eta(y))v_1\|_{L^\infty}=O(h^{1/2})\|v_1\|_{L^2}$; it remains to take $y=y_0$.

(6.7)*:* Follows immediately from (6.5), (6.6), (6.2), Cauchy-Riemann equations, and the fact that $\operatorname{Re}(\eta_0+i\tilde\omega_+)<0$. □

FIGURE 3. A contour with horizontal control

If $\tilde{P}_x$ were a semiclassical Schrödinger operator with a strictly positive potential, then a standard integration by parts argument would give us $\|u\|_{L^2} \leq C\|\tilde{P}_x u\|_{L^2}$ on any interval for each function u satisfying the condition (6.7) at the right endpoint of this interval and the opposite condition at its left endpoint. We now generalize this argument to our case. Assume that we have vertical control at the points $x_\pm$, $\pm x_\pm > X_0$, and let $(y_\pm, \eta_\pm)$ be the corresponding microlocalization points. Let γ be a contour in the z plane; we say that we have **horizontal control** on γ if:

- $\gamma \cap \{|\operatorname{Re} z| \leq X_0\} \subset \mathbb{R}$;
- the endpoints of γ are $z_\pm = x_\pm + iy_\pm$;
- γ is given by $\operatorname{Im} z = F(\operatorname{Re} z)$, where F is a smooth function and $F'(x_\pm) = 0$;
- $\operatorname{Re}[(1 + iF'(x))\widetilde{V}_x(x + iF(x))] \geq \frac{1}{C_1} > 0$ for all x.

Now, let u be as in the beginning of this section and define $u_\gamma(x)$, $x_- \leq x \leq x_+$, by Proposition 4.4. Then $\tilde{P}_\gamma u_\gamma = f$, where

$$\tilde{P}_\gamma = \left(\frac{1}{1 + iF'(x)} hD_x\right)^2 + \widetilde{V}_x(x + iF(x)).$$

If we have vertical control at the endpoints of γ, then by (6.7),

$$\pm \operatorname{Re}(\overline{u_\gamma(x_\pm)} h\partial_x u_\gamma(x_\pm)) \leq -|u_\gamma(x_\pm)|^2/C < 0.$$

Now, assume that we have horizontal control on γ. Then we can integrate by parts to get

$$\begin{gathered}\int_{x_-}^{x_+} \operatorname{Re}(\overline{u_\gamma}(1 + iF'(x))f)\,dx = \int_{x_-}^{x_+} \operatorname{Re}(\overline{u_\gamma} \cdot (1 + iF'(x))\tilde{P}_\gamma u_\gamma)\,dx \\ = \int_{x_-}^{x_+} \operatorname{Re} \frac{|hD_x u_\gamma|^2}{1 + iF'(x)}\,dx + \int_{x_-}^{x_+} \operatorname{Re}[(1 + iF'(x))\widetilde{V}_x(x + iF(x))] \cdot |u_\gamma|^2\,dx \\ -h^2 \operatorname{Re}(\overline{u_\gamma}\partial_x u_\gamma)|_{x=x_-}^{x_+} \geq \frac{1}{C_1}(\|u_\gamma\|_{L^2}^2 + h(|u_\gamma(x_+)|^2 + |u_\gamma(x_-)|^2)).\end{gathered} \tag{6.9}$$

Therefore,

$$\|u_\gamma\|_{L^2} \leq C\|f\|_{L^2}.$$

It follows that the operator R_x from (4.8) is correctly defined and

$$\|1_{K_x} R_x 1_{K_x}\|_{L^2\to L^2} \leq Ch^2,$$

This proves the estimate (3.8) under the assumptions made above.

We now prove (3.9). We concentrate on the estimate on K_+; the case of K_- is considered in a similar fashion. First of all, it follows from (6.9) that

$$|u_\gamma(x_+)| \leq Ch^{-1/2}\|f\|_{L^2}. \tag{6.10}$$

Now, assume that we have vertical control at every point of the interval $I_+ = [x_+, x_+ + 1]$ and let $(y(x), \eta(x))$, $x \in I_+$, be the corresponding microlocalization points. Let $v_x(z) = e^{-i\omega_+ z}u(z)$ and put $v_2(x) = v_x(x + iy(x))$; then

$$|v_2(x_+)| \leq Ce^{\operatorname{Im}(\omega_+ z_+)}|u_\gamma(x_+)|. \tag{6.11}$$

Now, by Proposition 6.1, we have

$$|h\partial_x \ln |v_2(x)| - \eta(x)| \leq Ch^{3/4},\ x \in I_+. \tag{6.12}$$

Integrating (6.12) and combining it with (6.10) and (6.11), we see that if

$$\operatorname{Im}(\tilde\omega_+ z_+) + \int_{x_+}^{x_+ + 1} \eta(x)\, dx < -2\delta_0 \tag{6.13}$$

for some $\delta_0 > 0$, then $|v_2(x_+ + 1)| \leq Ce^{-\delta_0/h}\|f\|_{L^2}$. Next, $v_2(x_+ + 1)$ is the value of v at the microlocalization point; therefore, by Proposition 6.1 and (5.1),

$$\sup_{y\in\mathbb{R}} |v_x(x_+ + 1 + iy)| \leq Ch^{-1/4}e^{-\delta_0/h}\|f\|_{L^2}.$$

Finally, recall that $v_x(z) = v_w(e^{-A_+ z})$, where the function $v_w(w)$ is holomorphic inside the disc $B_w = \{|w| \leq e^{-A_+(x_+ + 1)}\}$. The change of variables $w \to r$ is holomorphic by Proposition 4.1; let K_+^w be the image of K_+ under this change of variables. If δ_{r0} is small enough, then K_+^w lies in the interior of B_w; then by the maximum principle and Cauchy estimates on derivatives, we can estimate $\|v_w\|_{C^N(K_+^w)}$ for each N by $O(h^\infty)\|f\|_{L^2}$. This completes the proof of (3.9) if the conditions above are satisfied.

To prove part 2 of Proposition 3.2, it remains to establish both vertical and horizontal control in our situation:

Proposition 6.2. *Assume that $\delta_r > 0$. Then there exist ε_r and $x_\pm$, $\pm x_\pm > X_0$, such that under the conditions* (6.1)*,*

- *we have vertical control at every point of the intervals $I_+ = [x_+, x_+ + 1]$ and $I_- = [x_- - 1, x_-]$;*
- *we have horizontal control on a certain contour γ;*
- *the inequality* (6.13) *(and its analogue on I_-) holds.*

Proof. Let us first assume that $a\tilde{k} = \tilde{\mu} = \tilde{\omega} = 0$. Then $\tilde{\omega}_\pm = 0$ and $q(y,\eta) = -\eta^2 + \Delta_r(x_+ + iy)$. Therefore, if we choose x_+ large enough, there exists exactly one solution (y_0, η_0) to the equation $q(y,\eta) = 0$ with $\eta \leq 0$, and this solution has $y_0 = 0$. It is easy to verify that in that case we have vertical control on I_+. Similarly one can choose the point x_-; moreover, we can assume that $K_r \subset (x_-, x_+)$ after the change of variables $r \to x$. Next, since $\widetilde{V}_x = \Delta_r$, we can take γ to be the interval $[x_-, x_+]$ of the real line. The condition (6.13) holds because $\eta(x) < 0$ for every x and $\tilde{\omega}_\pm = 0$.

Now, fix $x_\pm$ as above. The parameters of our problem are a, varying in a compact set, Λ and M, both fixed, and $a\tilde{k}, \tilde{\mu}, \tilde{\omega}$. By the implicit function theorem, if the last three parameters are small enough, the (open) conditions of vertical control and the condition (6.13) are still satisfied, yielding $y_\pm$ close to zero. Then one can take the contour γ defined by $\operatorname{Im} z = F(\operatorname{Re} z)$, where $F = 0$ near K_r, $F(x_\pm) = y_\pm$, and F is small in C^∞. For small values of $a\tilde{k}, \tilde{\mu}, \tilde{\omega}$, we will still have horizontal control on this γ, proving the proposition. □

7. Resonance free strip

In this section, we prove Theorem 5. First of all, by Proposition 3.4, it suffices to prove

Proposition 7.1. *Fix $\delta_r > 0$, $\varepsilon_e > 0$, and a large constant C'. Then there exist constants $a_0 > 0$ and C'' such that for*

$$|\operatorname{Re}\lambda| + k^2 \leq C'|\operatorname{Re}\omega|^2,\ |a| < a_0,\ |\operatorname{Re}\omega| \geq 1/C'',$$
$$|\operatorname{Im}\omega| \leq 1/C'',\ |\operatorname{Im}\lambda| \leq |\operatorname{Re}\omega|/C''$$

we have

$$\|1_{K_r} R_r(\omega,\lambda,k) 1_{K_r}\|_{L^2\to L^2} \leq C''|\omega|^{\varepsilon_e - 1}.$$

Indeed, we take C' large enough so that $C_k^2(1+|\omega|)^2 + L \leq C'|\omega|^2/2$; then, we put $l_1 = L$ and $l_2 = |\operatorname{Re}\omega|/C''$.

Next, we reformulate Proposition 7.1 in semiclassical terms. Without loss of generality, we may assume that $\operatorname{Re}\omega > 0$. Put $h = (\operatorname{Re}\omega)^{-1}$ and consider the rescaled operator

$$\tilde{P}_x = h^2 P_x = h^2 D_x^2 + (\tilde{\lambda} + ih\tilde{\mu})\Delta_r - (1+\alpha)^2((r^2+a^2)(1+ih\nu) - a\tilde{k})^2.$$

Here P_x is the operator in (4.2) and

$$\tilde{\lambda} = h^2 \operatorname{Re}\lambda,\ \tilde{k} = hk,\ \tilde{\mu} = h\operatorname{Im}\lambda,\ \nu = \operatorname{Im}\omega.$$

Then it suffices to prove that for h small enough and under the conditions

$$|\tilde{\lambda}| \leq C',\ |\tilde{k}| \leq C',\ |\tilde{\mu}| \leq 1/C'',\ |\nu| \leq 1/C'', \tag{7.1}$$

for each $f(x) \in L^2 \cap \mathcal{E}'(K_x)$ and solution $u(x)$ to the equation $\tilde{P}_x u = f$ which is outgoing in the sense of Definition 4.1, we have

$$\|u\|_{L^2(K_x)} \leq Ch^{-1-\varepsilon_e}\|f\|_{L^2}. \tag{7.2}$$

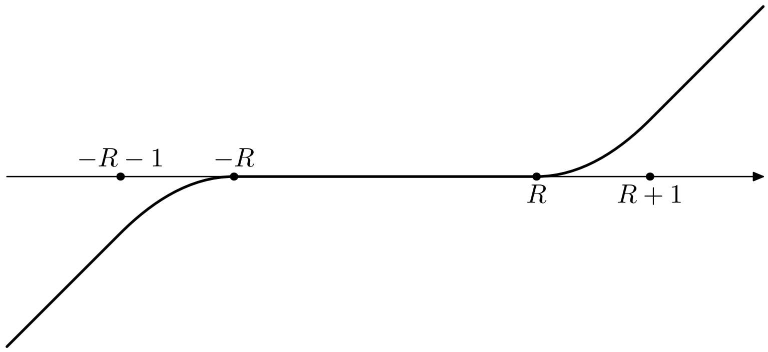

FIGURE 4. The contour used for complex scaling

(Here K_x is the image of $K_r = (r_- + \delta_r, r_+ - \delta_r)$ under the change of variables $r \to x$.) We write $\tilde{P}_x = h^2 D_x^2 + \widetilde{V}_0 + ih\widetilde{V}_1$, where

$$\begin{gathered}\widetilde{V}_0 = \tilde{\lambda}\Delta_r - (1+\alpha)^2(r^2 + a^2 - a\tilde{k})^2,\\ \widetilde{V}_1 = \tilde{\mu}\Delta_r - \nu(1+\alpha)^2(r^2+a^2)((r^2+a^2)(2+ih\nu) - 2a\tilde{k})^2.\end{gathered}$$

We note that $\widetilde{V}_0$ is real-valued and $\|\widetilde{V}_1\|_{L^\infty} \leq C/C''$ for some global constant C.

We now apply the method of complex scaling. (This method was first developed by Aguilar and Combes in [1]; see [32] and the references there for more recent developments.) Consider the contour γ in the complex plane given by $\operatorname{Im} x = F(\operatorname{Re} x)$, with F defined by

$$F(x) = \begin{cases} 0, & |x| \leq R; \\ F_0(x-R), & x \geq R; \\ -F_0(-x-R), & x \leq -R. \end{cases} \tag{7.3}$$

Here $R > X_0$ is large and $F_0 \in C_0^\infty(0,\infty)$ is a fixed function such that $F_0' \geq 0$ and $F_0'' \geq 0$ for all x and $F_0'(x) = 1$ for $x \geq 1$. (We could use a contour which forms an arbitrary fixed angle $\tilde{\theta} \in (0, \pi/2)$ with the horizonal axis for large x; we choose the angle $\pi/4$ to simplify the formulas.) Now, let u be an outgoing solution to the equation $\tilde{P}_x u = f \in L^2 \cap \mathcal{E}'(K_x)$, as above. By Proposition 4.4, we can define the restriction u_γ of u to γ and $\tilde{P}_\gamma u_\gamma = f$, where

$$\tilde{P}_\gamma = \Big(\frac{h}{1+iF'(x)} D_x\Big)^2 + \widetilde{V}_0(x + iF(x)) + ih\widetilde{V}_1(x+iF(x)).$$

Also, for a and h small enough, u_γ lies in $H^2(\mathbb{R})$. Therefore, in order to prove (7.2), it is enough to show that for each $u_\gamma \in H^2(\mathbb{R})$, we have

$$\|u_\gamma\|_{L^2(\mathbb{R})} \leq Ch^{-1-\varepsilon_e}\|\tilde{P}_\gamma u_\gamma\|_{L^2(\mathbb{R})}. \tag{7.4}$$

Let p_0 and $p_{\gamma 0}$ be the semiclassical principal symbols of $\tilde{P}_x$ and $\tilde{P}_\gamma$:

$$\begin{aligned} p_0(x,\xi) &= \xi^2 + \widetilde{V}_0(x), \\ p_{\gamma 0}(x,\xi) &= \frac{\xi^2}{(1+iF'(x))^2} + \widetilde{V}_0(x+iF(x)). \end{aligned}$$

The key property of the operator $\tilde{P}_\gamma$, as opposed to $\tilde{P}_x$, is ellipticity at infinity, which follows from the fact that $\widetilde{V}_0(\pm\infty) = -\tilde{\omega}_{0\pm}^2$, where

$$\tilde{\omega}_{0\pm} = (1+\alpha)(r_\pm^2 + a^2 - a\tilde{k}) \geq 1/C > 0$$

if a is small enough. Certain other properties of the symbol $p_{\gamma 0}$ can be derived using only the behavior of $\widetilde{V}_0$ near infinity given by (4.4); we state them for a general class of potentials:

Proposition 7.2. *Assume that $V(x)$, $x > 0$, is a real-valued potential such that for $x > X_0$, we have $V(x) = V_+(e^{-A_+ x})$ for a certain constant $A_+ > 0$ and a function $V_+(w)$ holomorphic in $\{|w| < e^{-A_+ X_0}\}$; assume also that $V_+(0) < 0$. Let $F(x)$ be as in* (7.3)*, for $R > X_0$, and put*

$$\begin{aligned} p(x,\xi) &= \xi^2 + V(x), \\ p_\gamma(x,\xi) &= \frac{\xi^2}{(1+iF'(x))^2} + V(x+iF(x)). \end{aligned}$$

Then there exists a constant C_c such that for R large enough and $\delta > 0$ small enough,

$$\text{if } x \geq R+1, \text{ then } |p_\gamma(x,\xi)| \geq 1/C_c > 0, \tag{7.5}$$

$$\text{if } |p_\gamma(x,\xi)| \leq e^{-A_+ R}, \text{ then } \operatorname{Im} p_\gamma(x,\xi) \leq 0, \tag{7.6}$$

$$\text{if } |p_\gamma(x,\xi)| \leq \delta, \text{ then } |p(x,\xi)| \leq C_c\delta,\ |\nabla(\operatorname{Re} p_\gamma - p)(x,\xi)| \leq C_c\delta. \tag{7.7}$$

Similar facts hold if V is defined on $x < 0$ instead.

Proof. Without loss of generality, we assume that $A_+ = 1$ and $V_+(0) = -1$. First of all, if $x \geq R+1$, then

$$p_\gamma(x,\xi) = -i\xi^2/2 + V(x+iF(x)) = -i\xi^2/2 - 1 + O(e^{-R}).$$

For R large enough, we then get $|p_\gamma(x,\xi)| \geq 1/2$, thus proving (7.5).

For the rest of the proof, we may assume that $R \leq x \leq R+1$. Then, since F_0' is increasing, we get $0 \leq F(x) \leq F'(x)$. Suppose that $|p_\gamma(x,\xi)| \leq \delta$; then

$$\begin{aligned} \frac{\xi^2}{(1+iF'(x))^2} &= -V(x+iF(x)) + O(\delta) \\ &= -V(x)(1+O(\delta+e^{-R}F(x))) = 1 + O(\delta + e^{-R}). \end{aligned} \tag{7.8}$$

Taking the arguments of both sides, we get

$$F'(x) \leq C(\delta + e^{-R}F(x)) \leq C\delta + Ce^{-R}F'(x).$$

Then for R large enough,

$$|p_\gamma(x,\xi)| \leq \delta \to F'(x) \leq C\delta.$$

This proves (7.7), if we note that F'' is bounded and

$$\operatorname{Re} p_\gamma(x,\xi) - p(x,\xi) = \xi^2 G_1(F'(x)^2) + G_2(F(x),x)$$

for certain smooth functions G_1 and G_2 that are equal to zero at $F' = 0$ and $F = 0$, respectively.

Now, putting $\delta = e^{-R}$ and taking the arguments and then the absolute values of both sides of (7.8), we get for $|p_\gamma| \leq \delta$,

$$F'(x) = O(e^{-R}),\ \xi^2 = 1 + O(e^{-R}).$$

Therefore,

$$\operatorname{Im} p_\gamma(x,\xi) = -2F'(x) + O(e^{-R}(F(x) + F'(x))) = F'(x)(-2 + O(e^{-R})),$$

which proves (7.6). □

Now, we study the trapping properties of the Hamiltonian flow of p_0 at the zero energy:

Proposition 7.3. *There exist constants C_V and δ_V such that for a small enough and every $\tilde\lambda$, $\tilde k$ satisfying* (7.1)*, at least one of the three dynamical cases below holds:*

1. $\widetilde{V}_0 \leq -\delta_V$ *everywhere;*
2. $\{|\widetilde{V}_0| \leq \delta_V\} = [x_1,x_2] \sqcup [x_3,x_4]$*, where* $-C_V \leq x_1 < x_2 < x_3 < x_4 \leq C_V$ *and* $\widetilde{V}_0' \geq 1/C_V$ *on* $[x_1,x_2]$*,* $\widetilde{V}_0' \leq -1/C_V$ *on* $[x_3,x_4]$*;*
3. $\{\widetilde{V}_0 \geq -\delta_V\} = [x_1,x_2]$ *with* $|x_j| \leq C_V$*,* $\widetilde{V}_0'' \leq -1/C_V$ *on* $[x_1,x_2]$*.*

Proof. First of all, if $\tilde\lambda$ is small enough or $\tilde\lambda < 0$, then we have $\widetilde{V}_0 < 0$ everywhere and therefore case (1) holds for δ_V small enough. Therefore, we may assume that $1/C \leq \tilde\lambda \leq C$ for some constant C. Now, we write

$$\begin{aligned}
\widetilde{V}_0(x) &= G_V(r)(F_V(r) - \tilde\lambda^{-1}),\\
G_V(r) &= \tilde\lambda(1+\alpha)^2(r^2+a^2-a\tilde k)^2,\\
F_V(r) &= \frac{\Delta_r}{(1+\alpha)^2(r^2+a^2-a\tilde k)^2}.
\end{aligned}$$

Note that $1/C \leq G_V(r) \leq C$ for a small enough, some constant C, and all r. As for F_V, there exists $\varepsilon > 0$ such that for a small enough, $\partial_r F_V(r) \geq 1/C > 0$ for $r \leq 3M - \varepsilon$, $\partial_r F_V(r) \leq -1/C < 0$ for $r \geq 3M + \varepsilon$, and $\partial_r^2 F_V(r) \leq -1/C < 0$ for $|r - 3M| \leq \varepsilon$. Indeed, this is true for $a = 0$ and follows for small a by a perturbation argument. Let $r_0 \in [3M - \varepsilon, 3M + \varepsilon]$ be the point where F_V achieves its maximal value. Take small $\delta_1 > 0$; then we have one of the following three cases, each of which in turn implies the corresponding case in the statement of this proposition:

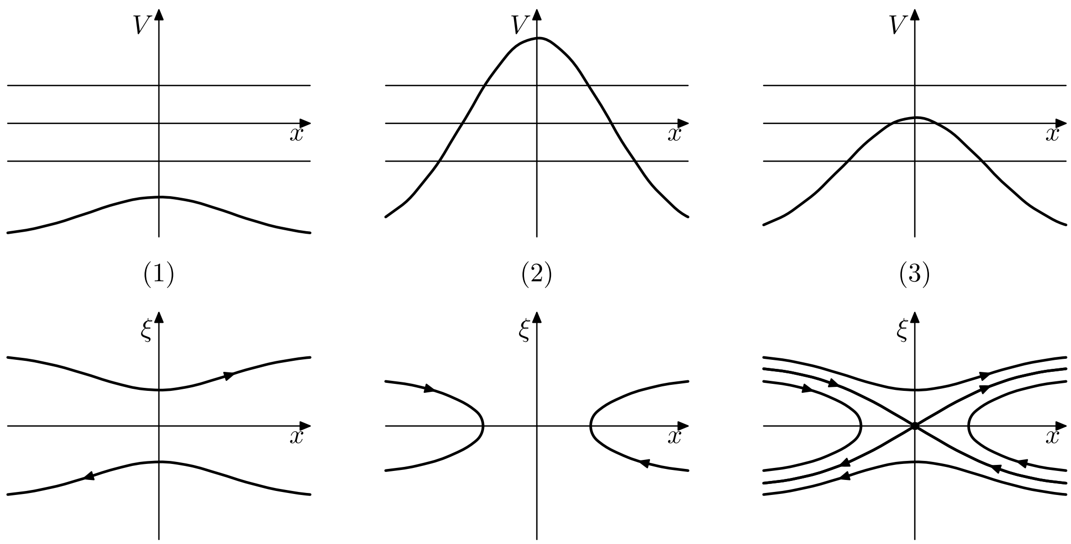

FIGURE 5. Three cases for the potential $\widetilde{V}_0$ and its Hamiltonian flow near the zero energy. The horizontal lines correspond to $\widetilde{V}_0 = \pm\delta_V$.

(1) $F_V(r_0) - \tilde{\lambda}^{-1} \leq -\delta_1$. Then $\widetilde{V}_0(x) < -\delta_V$ for all x and $\delta_V > 0$ small enough.
(2) $F_V(r_0) - \tilde{\lambda}^{-1} \geq \delta_1$. Then for $\delta_2 < \delta_1/2$, $\{|F_V - \tilde{\lambda}^{-1}| \leq \delta_2\} = [x_1, x_2] \sqcup [x_3, x_4]$, where $x_2 < x_3$, x_j are bounded by a global constant (since $\tilde{\lambda}$ is bounded from above), and $\partial_r F_V(r) > 1/C_\delta > 0$ for $x \in [x_1, x_2]$, $\partial_r F_V(r) < -1/C_\delta < 0$ for $x \in [x_3, x_4]$. Here C_δ is a constant depending on δ_1, but not on δ_2. It follows that for δ_2 small enough depending on δ_1, we have $\widetilde{V}_0'(x) > 0$ for $x \in [x_1, x_2]$ and $\widetilde{V}_0'(x) < 0$ for $x \in [x_3, x_4]$; also, for δ_V small enough, we have $\{|\widetilde{V}_0| \leq \delta_V\} \subset [x_1, x_2] \sqcup [x_3, x_4]$.
(3) $|F_V(r_0) - \tilde{\lambda}^{-1}| < \delta_1$. Then $\{F_V - \tilde{\lambda}^{-1} > -\delta_1\} = [x_1, x_2]$ with $\partial_r^2 F_V(r) < -1/C < 0$ for $x \in [x_1, x_2]$. For δ_1 small enough, we then get $\widetilde{V}_0'' < -1/C < 0$ for $x \in [x_1, x_2]$, and for δ_V small enough, we have $\{\widetilde{V}_0 \geq -\delta_V\} \subset [x_1, x_2]$. □

We are now ready to prove (7.4) and, therefore, Theorem 5. Fix R large enough so that Proposition 7.2 holds. The first two cases are in Proposition 7.3 are nontrapping; it follows that there exists an escape function $G \in C_0^\infty(\mathbb{R}^2)$ such that $H_{p_0}G < 0$ on $\{|p_0| \leq \delta_V/2\} \cap \{|x| \leq R+2\}$. In the third case, we have hyperbolic trapping with the trapped set consisting of a single point $(x_0, 0)$, where x_0 is the point where $\widetilde{V}_0$ achieves its maximal value; therefore, there still exists an escape function $G \in C_0^\infty(\mathbb{R}^2)$ such that $H_{p_0}G \leq 0$ on $\{|p_0| \leq \delta_V/2\} \cap \{|x| \leq R+2\}$ and $H_{p_0}G < 0$ on $\{|p_0| \leq \delta_V/2\} \cap \{|x| \leq R+2\} \setminus U(x_0, 0)$, where U is a neighborhood of $(x_0, 0)$ which can be made arbitrarily small by the choice of G (see [18, Proposition A.6]). Now, given Proposition 7.2, we can choose $\delta_0 > 0$ such that

$$\operatorname{Im} p_{\gamma 0} \leq 0 \text{ on } \{|p_{\gamma 0}| \leq \delta_0\} \tag{7.9}$$

and for cases (1) and (2) of Proposition 7.3, we have

$$H_{\operatorname{Re} p_{\gamma 0}} G \leq -1/C < 0 \text{ on } \{|p_{\gamma 0}| \leq \delta_0\}, \tag{7.10}$$

and for case (3) of Proposition 7.3, we have

$$\begin{gathered} H_{\operatorname{Re} p_{\gamma 0}} G \leq 0 \text{ on } \{|p_{\gamma 0}| \leq \delta_0\}, \\ H_{\operatorname{Re} p_{\gamma 0}} G \leq -1/C < 0 \text{ on } \{|p_{\gamma 0}| \leq \delta_0\} \setminus U(x_0, 0). \end{gathered} \tag{7.11}$$

Armed with these inequalities, we can handle the nontrapping cases even without requiring that μ and ν be small. The statement below follows the method initially developed in [27] and is a special case of the results in [14, Chapter 6]; however, we choose to present the proof in our simple case:

Proposition 7.4. *Assume that either case (1) or case (2) of Proposition 7.3 holds. Then for $\tilde\lambda$ and $\tilde k$ bounded by C', $\tilde\mu$ and ν bounded by some constant, and h small enough, we have*

$$\|u_\gamma\|_{L^2} \leq Ch^{-1} \|\tilde P_\gamma u_\gamma\|_{L^2} \tag{7.12}$$

for each $u_\gamma \in H^2(\mathbb{R})$.

Proof. Take $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\operatorname{supp}\chi \subset \{|p_{\gamma 0}| < \delta_0\}$, but $\chi = 1$ near $\{p_{\gamma 0} = 0\}$. Next, take $s > 0$, to be chosen later, and put

$$\tilde P_{\gamma,s} = e^{sG^w} \tilde P_\gamma e^{-sG^w},\ u_{\gamma,s} = e^{sG^w} \chi^w u_\gamma.$$

Take $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ supported in $\{|p_{\gamma 0}| < \delta_0\}$, but such that $\chi_1 = 1$ near $\operatorname{supp}\chi$. Then by part 1 of Proposition 5.3 and (5.2),

$$\|(1 - \chi_1^w) u_{\gamma,s}\| = O(h^\infty) \|u_\gamma\|. \tag{7.13}$$

(In the proof of the current proposition, as well as the next one, we only use L^2 norms.) Also, for some s-dependent constant C,

$$C^{-1} \|\chi^w u_\gamma\| \leq \|u_{\gamma,s}\| \leq C \|u_\gamma\|.$$

Now, by part 2 of Proposition 5.3, we have

$$\tilde P_{\gamma,s} = \tilde P_{\gamma 0} + ihV_1 + ish(H_{p_{\gamma,0}} G)^w + O(h^2).$$

Here $\tilde P_{\gamma 0}$ is the principal part of $\tilde P_\gamma$ (without V_1) and the constant in $O(h^2)$ depends on s. We then have

$$\begin{gathered} \operatorname{Im}(\tilde P_{\gamma,s} \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) = \operatorname{Im}(\tilde P_{\gamma 0} \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) + h \operatorname{Re}(V_1 \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) \\ + sh((H_{\operatorname{Re} p_{\gamma,0}} G)^w \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) + O(h^2) \|\chi_1^w u_{\gamma,s}\|^2. \end{gathered}$$

By (7.9) and part 1 of Proposition 5.2,

$$\operatorname{Im}(\tilde P_{\gamma 0} \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) \leq Ch \|\chi_1^w u_{\gamma,s}\|^2 + O(h^\infty) \|u_\gamma\|^2.$$

Next, by (7.10) and part 2 of Proposition 5.2,

$$((H_{\operatorname{Re} p_{\gamma,0}} G)^w \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) \leq -C^{-1} \|\chi_1^w u_{\gamma,s}\|^2 + O(h^\infty) \|u_\gamma\|^2.$$

Adding these up, we get

$$\operatorname{Im}(\tilde{P}_{\gamma,s}\chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) \leq -h(C_1^{-1}s - C_1 - O(h))\|\chi_1^w u_{\gamma,s}\|^2 + O(h^\infty)\|u_\gamma\|^2.$$

Here the constants in $O(\cdot)$ depend on s, but the constant C_1 does not. Therefore, if we choose s large enough and h-independent, then for small h we have the estimate

$$\|\chi_1^w u_{\gamma,s}\|^2 \leq Ch^{-1}\|\tilde{P}_{\gamma,s}\chi_1^w u_{\gamma,s}\| \cdot \|\chi_1^w u_{\gamma,s}\| + O(h^\infty)\|u_\gamma\|^2.$$

Together with (7.13), this gives

$$\|\chi^w u_\gamma\|^2 \leq Ch^{-1}\|\tilde{P}_\gamma u_\gamma\| \cdot \|u_\gamma\| + Ch^{-1}\|[\tilde{P}_\gamma, \chi^w]u_\gamma\| \cdot \|u_\gamma\| + O(h^\infty)\|u_\gamma\|^2.$$

Applying Proposition 5.1 to estimate $(1-\chi^w)u_\gamma$ and the commutator term above, we get the estimate (7.12). $\square$

Remark. The method described above can actually be used to obtain a **logarithmic** resonance free region; however, since we expect the resonances generated by trapping to lie asymptotically on a lattice as in [31], we only go a fixed amount deep into the complex plane.

The third case in Proposition 7.3 is where trapping occurs, and we analyse it as in [38]: (See also [10] for a different method of solving the same problem.)

Proposition 7.5. *Assume that case (3) in Proposition 7.3 holds, and fix $\varepsilon_e > 0$. Then for $\tilde{\lambda}$ and $\tilde{k}$ bounded by C' and for $\tilde{\mu}, \nu, h$ small enough, we have*

$$\|u_\gamma\|_{L^2} \leq Ch^{-1-\varepsilon_e}\|\tilde{P}_\gamma u_\gamma\|_{L^2} \tag{7.14}$$

for each $u_\gamma \in H^2(\mathbb{R})$.

Proof. First, we establish [38, Lemma 4.1] in our case. Let x_0 be the point where $\widetilde{V}_0$ achieves its maximum value. We may assume that $|p_0(x_0,0)| = |\widetilde{V}_0(x_0)| < \delta_0/2$; otherwise, we are in one of the two nontrapping cases. Put

$$\tilde{\xi}(x) = \operatorname{sgn}(x-x_0)\sqrt{\widetilde{V}_0(x_0) - \widetilde{V}_0(x)};$$

since $\widetilde{V}_0''(x_0) < 0$, it is a smooth function. Then, define the functions $\varphi_\pm(x,\xi) = \xi \mp \tilde{\xi}(x)$. We have

$$H_{p_0}\varphi_\pm(x,\xi) = \mp c(x,\xi)\varphi_\pm(x,\xi),$$

where $c(x,\xi) = 2\partial_x\tilde{\xi}(x)$ is greater than zero near the trapped point $(x_0,0)$. Also, $\{\varphi_+,\varphi_-\} = c(x,\xi)$. Next, take $\tilde{h} > h$ and large $C_0 > 0$, let $\chi_0 \geq 0$ be supported in a small neighborhood of $(x_0,0)$ with $\chi_0 = 1$ near this point, and define the modified escape function [38, (4.6)]

$$G_1(x,\xi) = -\chi_0(x,\xi)\log\frac{\varphi_-^2(x,\xi) + h/\tilde{h}}{\varphi_+^2(x,\xi) + h/\tilde{h}} + C_0\log(1/h)G(x,\xi).$$

Here G is an escape function satisfying (7.11). We can write

$$\begin{aligned} H_{\operatorname{Re} p_{\gamma,0}} G_1 = -2\chi_0 c\Big(\frac{\varphi_-^2}{\varphi_-^2 + h/\tilde h} + \frac{\varphi_+^2}{\varphi_+^2 + h/\tilde h}\Big) \\ -(H_{p_0}\chi_0)\log\frac{\varphi_-^2 + h/\tilde h}{\varphi_+^2 + h/\tilde h} + C_0\log(1/h) H_{\operatorname{Re} p_{\gamma 0}} G(x,\xi). \end{aligned} \tag{7.15}$$

Take χ_1 supported in $\{|p_{\gamma 0}| < \delta_0\}$, but equal to 1 near $\{p_{\gamma 0} = 0\}$. Then one can use the uncertainty principle [38, Section 4.2] to show that if χ_2 is supported inside $\{\chi_0 = 1\}$, but $\chi_2 = 1$ near $(x_0, 0)$, then for each $v \in L^2$,

$$\begin{aligned} ((H_{\operatorname{Re} p_{\gamma,0}} G_1)^w \chi_1^w v, \chi_1^w v) &\leq (-C^{-1}\tilde h + O(\tilde h^2))\|\chi_2^w v\|^2 + O(\log(1/h))\|(1-\chi_2^w)\chi_1^w v\|^2 \\ &\quad - C_0 C^{-1}\log(1/h)\|(1-\chi_2^w)\chi_1^w v\|^2 + O(C_0 h\log(1/h))\|\chi_1^w v\|^2 + O(h^\infty)\|v\|^2 \\ &\leq -(C^{-1}\tilde h - O(\tilde h^2 + C_0 h\log(1/h)))\|\chi_2^w v\|^2 \\ &\quad -(C_0 C^{-1}\log(1/h) - O(C_0 h\log(1/h) + \log(1/h)))\|(1-\chi_2^w)\chi_1^w v\|^2 + O(h^\infty)\|v\|^2. \end{aligned}$$

If we fix C_0 large enough and $\tilde h$ small enough and assume that h small enough, then

$$((H_{\operatorname{Re} p_{\gamma,0}} G_1)^w \chi_1^w v, \chi_1^w v) \leq -C^{-1}\log(1/h)\|(1-\chi_2^w)\chi_1^w v\|^2 - C^{-1}\tilde h\|\chi_1^w v\|^2 + O(h^\infty)\|v\|^2.$$

Next, we conjugate by exponential pseudodifferential weights. First of all, one can prove that

$$\|G_1^w\|_{L^2\to L^2} \leq C\log(1/h);$$

therefore,

$$\|e^{sG_1^w}\|_{L^2\to L^2} \leq h^{-C|s|}.$$

Let χ be supported in $\{\chi_1 = 1\}$, but $\chi = 1$ near $\{p_{\gamma 0} = 0\}$, and

$$P_{\gamma,s} = e^{sG_1^w} P_\gamma e^{-sG_1^w},\ u_{\gamma,s} = e^{sG_1^w}\chi^w u_\gamma;$$

then [38, Section 4.3]

$$P_{\gamma,s} = P_\gamma + ish(H_{p_{\gamma 0}} G_1)^w + O(s^2\tilde h h + s h^{3/2}\tilde h^{3/2} + h^2).$$

Therefore, since $\operatorname{Im} p_{\gamma 0} = 0$ near $\operatorname{supp}\chi_2$,

$$\begin{aligned} \operatorname{Im}(\tilde P_{\gamma,s}\chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) &= \operatorname{Im}(\tilde P_{\gamma 0}\chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) + h\operatorname{Re}(V_1\chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) \\ &\quad + sh\operatorname{Re}((H_{\operatorname{Re} p_{\gamma,0}} G_1)^w \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) + O(s^2 h\tilde h + s h^{3/2}\tilde h^{3/2} + h^2)\|\chi_1^w u_{\gamma,s}\|^2 \\ &\leq O(h)\|(1-\chi_2^w)\chi_1^w u_{\gamma,s}\|^2 + h\|V_1\|_{L^\infty}\|\chi_1^w u_{\gamma,s}\|^2 - C^{-1} sh\log(1/h)\|(1-\chi_2^w)\chi_1^w u_{\gamma,s}\|^2 \\ &\quad - C^{-1} sh\tilde h\|\chi_1^w u_{\gamma,s}\|^2 + O(s^2 h\tilde h + s h^{3/2}\tilde h^{3/2} + h^2)\|\chi_1^w u_{\gamma,s}\|^2 + O(h^\infty)\|u_\gamma\|^2. \end{aligned}$$

Here $\tilde{P}_{\gamma 0}$ is the principal part of $\tilde{P}_\gamma$, as before. If we choose s small enough independently of h, then for small h,

$$\operatorname{Im}(\tilde{P}_{\gamma,s}\chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) \leq -C_1 sh\log(1/h)\|(1-\chi_2^w)\chi_1^w u_{\gamma,s}\|^2$$
$$-h(C^{-1}s\tilde{h} - \|V_1\|_{L^\infty})\|\chi_1^w u_{\gamma,s}\|^2 + O(h^\infty)\|u_\gamma\|^2.$$

Now, $\|V_1\|_{L^\infty}$ can be made very small by choosing $\tilde{\mu}$ and ν small enough. Then, we get

$$\|\chi^w u_\gamma\|^2 \leq Ch^{-1-Cs}\|u_\gamma\| \cdot \|P_\gamma \chi^w u_\gamma\| + O(h^\infty)\|u_\gamma\|^2.$$

By proceeding as in the end of Proposition 7.4, we get (7.14), provided that s is small enough. □

Acknowledgements. I would like to thank Maciej Zworski for suggesting the problem, lots of helpful advice, and encouragement, and Kiril Datchev, Mihai Tohaneanu, Daniel Tataru, and Tobias Schottdorf for some very helpful discussions. I am also grateful for partial support from NSF grant DMS-0654436. Finally, I am especially thankful to an anonymous referee for many suggestions to improve the manuscript.

References

[1] J. Aguilar and J.M. Combes, *A class of analytic perturbations for one-body Schrödinger Hamiltonians,* Comm. Math. Phys **22**(1971), 269–279.

[2] L. Andersson and P. Blue, *Hidden symmetries and decay for the wave equation on the Kerr spacetime,* `arXiv:0908.2265`.

[3] A. Bachelot, *Gravitational scattering of electromagnetic field by Schwarzschild black hole,* Ann. Inst. H. Poincaré Phys. Théor. **54**(1991), 261–320.

[4] A. Bachelot, *Scattering of electromagnetic field by de Sitter–Schwarzschild black hole,* in *Non-linear hyperbolic equations and field theory.* Pitman Res. Notes Math. Ser. **253**, 23–35.

[5] A. Bachelot and A. Motet-Bachelot, *Les résonances d'un trou noir de Schwarzschild,* Ann. Inst. H. Poincaré Phys. Théor. **59**(1993), 3–68.

[6] M. Ben-Artzi and A. Devinatz, *Resolvent estimates for a sum of tensor products with applications to the spectral theory of differential operators,* J. d'Analyse Math. **43**(1983/4), 215–250.

[7] J.-P. Berenger, *A perferctly matched layer for the absorption of electromagnetic waves,* J. Comput. Phys. **114**(1994), 185–200.

[8] E. Berti, V. Cardoso, and A. Starinets, *Quasinormal modes of black holes and black branes,* Class. Quant. Grav. **26**(2009), 163001.

[9] P. Blue and J. Sterbenz, *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space,* Comm. Math. Phys. **268**(2006), 481–504.

[10] J.-F. Bony, S. Fujie, T. Ramond, and M. Zerzeri, *Spectral projection, residue of the scattering amplitude, and Schrödinger group expansion for barrier-top resonances,* `arXiv:0908.3444`.

[11] J.-F. Bony and D. Häfner, *Decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric,* Comm. Math. Phys. **282**(2008), 697–719.

[12] B. Carter, *Hamilton-Jacobi and Schrödinger separable solutions of Einstein's equations,* Comm. Math. Phys. **10**(1968), 280–310.

[13] M. Dafermos and I. Rodnianski, *Lectures on black holes and linear waves,* `arXiv:0811.0354v1`.

[14] K. Datchev, *Distribution of resonances for manifolds with hyperbolic ends,* doctoral dissertation, University of California, Berkeley, 2010, `http://math.berkeley.edu/~datchev/main.pdf`.

[15] R. Donninger, W. Schlag, and A. Soffer, *A proof of Price's Law on Schwarzschild black hole manifolds for all angular momenta,* arXiv:0908.4292.
[16] R. Donninger, W. Schlag, and A. Soffer, *On pointwise decay of linear waves on a Schwarzschild black hole background,* arXiv:0911.3179.
[17] L.C. Evans and M. Zworski, *Semiclassical analysis,* lecture notes, version 0.8, http://math.berkeley.edu/~zworski/semiclassical.pdf.
[18] C. Gérard and J. Sjöstrand, *Semiclassical resonances generated by a closed trajectory of hyperbolic type,* Comm. Math. Phys. **108**(1987), 391–421.
[19] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau, *Decay of solutions of the wave equation in the Kerr geometry,* Comm. Math. Phys. **264**(2006), 465–503.
[20] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau, *Erratum: "Decay of solutions of the wave equation in the Kerr geometry",* Comm. Math. Phys. **280**(2008), 563–573.
[21] C. Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds,* Duke. Math. J. **129**(2005), 1–37.
[22] L. Hörmander, *The Analysis of Linear Partial Differential Operators III,* Springer, 1994.
[23] N.K. Kofinti, *Scattering of a Klein–Gordon particle by a black hole,* Internat. J. Theoret. Phys. **23**(1984), 991–999.
[24] K.D. Kokkotas and B.G. Schmidt, *Quasi-normal modes of stars and black holes,* Living Rev. Relativity **2**(1999), http://www.livingreviews.org/lrr-1999-2.
[25] R.A. Konoplya and A. Zhidenko, *High overtones of Schwarzschild-de Sitter quasinormal spectrum,* JHEP **0406**, 037 (2004).
[26] R.A. Konoplya and A. Zhidenko, *Decay of a charged scalar and Dirac fields in the Kerr-Newman-de Sitter background,* Phys. Rev. D **76**, 084018 (2007).
[27] A. Martinez, *Resonance free domains for non globally analytic potentials,* Ann. Inst. H. Poincaré **3**(2002), 739–756.
[28] R.R. Mazzeo and R.B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature,* J. Funct. Anal. **75**(1987), 260–310.
[29] R.R. Mazzeo and A. Vasy, *Resolvents and Martin boundaries of product spaces,* Geom. Funct. Anal. **12**(2002), 1018–1079.
[30] R.B. Melrose, A. Sá Barreto, and A. Vasy, *Asymptotics of solutions of the wave equation on de Sitter–Schwarzschild space,* arXiv:0811.2229.
[31] A. Sá Barreto and M. Zworski, *Distribution of resonances for spherical black holes,* Math. Res. Lett. **4**(1997), 103–121.
[32] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles,* J. Amer. Math. Soc. **4**(1991), 729–769.
[33] D. Tataru and M. Tohaneanu, *Local energy estimate on Kerr black hole backgrounds,* arXiv:0810.5766.
[34] D. Tataru, *Local decay of waves on asymptotically flat stationary space-times,* arXiv:0910.5290.
[35] M. Taylor, *Partial Differential Equations I. Basic Theory,* Springer, 1996.
[36] S.A. Teukolsky, *Rotating black holes: separable wave equations for gravitational and electromagnetic pertrubations,* Phys. Rev. Lett. **29**(1972), 1114–1118.
[37] M. Tohaneanu, *Strichartz estimates on Kerr black hole backgrounds,* arXiv:0910.1545.
[38] J. Wunsch and M. Zworski, *Resolvent estimates for normally hyperbolic trapped sets,* arXiv:1003.4640v2.

E-mail address: dyatlov@math.berkeley.edu

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA